Fluctuations of the one-dimensional polynuclear growth model with external sources

T. Imamura * and T. Sasamoto †

*Department of Physics, Graduate School of Science,
University of Tokyo,
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

†Department of Physics, Tokyo Institute of Technology,
Oh-okayama 2-12-1, Meguro-ku, Tokyo 152-8551, Japan

Abstract

The one-dimensional polynuclear growth model with external sources at edges is studied. The height fluctuation at the origin is known to be given by either the Gaussian, the GUE Tracy-Widom distribution, or certain distributions called GOE^2 and F_0, depending on the strength of the sources. We generalize these results and show that the scaling limit of the multi-point equal time height fluctuations of the model are described by the Fredholm determinant, of which the limiting kernel is explicitly obtained. In particular we obtain two new kernels, describing transitions between the above one-point distributions. One expresses the transition from the GOE^2 to the GUE Tracy-Widom distribution or to the Gaussian; the other the transition from F_0 to the Gaussian. The results specialized to the fluctuation at the origin are shown to be equivalent to the previously obtained ones via the Riemann-Hilbert method.

[Keywords: polynuclear growth; KPZ universality class; random matrices; Tracy-Widom distribution; Airy process]

*e-mail: imamura@monet.phys.s.u-tokyo.ac.jp
†e-mail: sasamoto@stat.phys.titech.ac.jp

1 Introduction

Surface growth is an interesting phenomenon in nature [1]. In particular various shapes show up due to the interplay of non-linearity, fluctuation and boundary effects. In [2], Kardar, Parisi and Zhang proposed a non-linear stochastic differential equation called the KPZ equation. This equation, although now considered to be insufficient for the description of most realistic situation in nature, defines a universality class of the surface growth phenomenon and plays a prominent role in the theoretical study.

In the one dimensional case we can analyze the KPZ equation exactly. The roughness and dynamical exponents were obtained by the renormalization techniques [2] and the Bethe Ansatz method [3, 4]. Recently we have been obtaining more sophisticated information for the height fluctuation in the one-dimensional KPZ universality class; not only the exponents but also the height distribution itself have been obtained. For various probabilistic models belonging to the KPZ universality class [8–13], it has turned out that the height fluctuation is equivalent to the Tracy-Widom distribution [5,6], the distribution of the largest eigenvalue in some random matrix ensemble [7].

Among these models we focus on the polynuclear growth (PNG) model. For the PNG model the relation to the random matrix theory was first pointed out by Prähofer and Spohn [14]. They mapped a specific PNG model to the longest increasing subsequence problem in random permutations and then applied the Baik-Deift-Johansson theorem [15]. The dependence of the height fluctuation on the geometry of the model is also studied based on the works of Baik and Rains [16–18] in random permutations with symmetries. As a result the deep connection with the various universality classes in random matrix theory (RMT) have been revealed. For example, for the droplet growth in an infinite line, the height distribution can be described by the GUE Tracy-Widom distribution [14] whereas in a half-infinite line, the height fluctuation at the origin can be described by the GSE/GOE/Gaussian fluctuation according to the strength of the nucleation rate at the origin [19, 25]. On the other hand the GOE Tracy-Widom distribution represents the height fluctuation in a growth on a flat substrate [19, 20].

Next we broaden our field of view from the height fluctuation at one point to that over some region of the surface. In other words, we address the issue about the roughness of the surface. The spatial configuration of the height fluctuation is expected to converge to the universal stochastic process after a proper scaling when the space direction is treated as the time direction of the process. In general a stochastic process is characterized by a dynamical correlation function together with a fluctuation at one point. Thus if we try to understand the universal aspect of the spatial configuration of the fluctuation, we need information about the multi-point correlations of the height fluctuation.

The correlation between distinct points of the PNG model is closely related to the multi-matrix model. In [21, 22], the multi-point equal time height fluctuations of the one-dimensional PNG model was studied for a droplet initial condition in an infinite space. It was found that the correlation is described by the Airy process, which is the process of the largest eigenvalue in the Dyson's Brownian motion model [29] for GUE. This also appeared in the facet fluctuation in the crystal [23, 24]. In [25], similar quantities are

evaluated for a special value of the external source in case of droplet growth in half space. It was shown that the correlation is described the Dyson's Brownian motion model which has transition between GOE/GSE to GUE. The same process also appeared in the vicious walk problem [26].

In this paper we study the PNG model with external sources. The goal of this study is to understand the universal process characterizing the roughness of the surface for this model. There are mainly two different regions. One is the region where the edge effects are dominant and the other is the region where the bulk dynamics prevails. The statistics of height fluctuation obey the one dimensional Brownian motion near edges and the Airy process in the bulk. When seen from far away, the above two types of regions are separated by a well distinguishable point. But we can focus into a small region around this point, in which the edge effects and bulk dynamics are competing. We are especially interested in this region because there appear new processes describing the transition of the surface fluctuation.

For this purpose we obtain the Fredholm determinant representation of the multi-point correlation function in the bulk region, near edges and around the special points mentioned above. We summarize the results as Theorems 3.1, 4.1 and 5.1. In particular the Fredholm determinant expressing the correlation in the intermediate region appears in this paper for the first time. They describe the transition between GOE^2 and GUE/Gaussian, or F_0 and Gaussian, where GOE^2 means the distribution of the larger of the largest eigenvalues of two independent GOEs and F_0 is a certain probability distribution which has no interpretation in RMT [16–18, 27].

The paper is organized as follows. In the next section, we recall the definition of the model and some known facts are reviewed. In section 3, we give the description of the equal-time multi-point fluctuation for finite system. The asymptotic results for fixed values of parameters are also stated. In section 4, the transition near the GOE^2 fluctuation is discussed. In the following section 5, the transition around F_0 is studied. In both section 4 and 5 we also discuss the connection between Fredholm representation and the Riemann-Hilbert representation of Baik-Rains [27]. The last section is devoted to the conclusion.

2 Model and One-point Height Fluctuation

In this article we mainly consider the discrete PNG model with external sources studied previously in [19, 27]. First of all, we briefly explain the PNG model. The PNG model is a simple model of layer-by-layer growth [1]. The discrete version of the model consists of the following three rules.

1. **nucleation**: A nucleation with height k is generated according to the geometric distribution. An object made by this rule is called a step.

2. **lateral growth of a step**: Once a step is produced, it grows laterally by one step in both directions during each time step.

3. **unification of steps**: When two steps with distinct heights collide, the height in the colliding point becomes that of the higher step.

Note that the rule 1 is only probabilistic and the last two rules are deterministic. These rules are illustrated in Fig.1.

We can formulate the above rules of time evolution mathematically as follows. Let $r \in \mathbb{Z}$ and $t \in \mathbb{N} = \{0, 1, 2, \cdots\}$ denote the discrete space and time coordinates respectively and $h(r,t)$ the height of the surface at position r and at time t. The rules $1 \sim 3$ can be collected as

$$h(r, t+1) = \max(h(r-1,t), h(r,t), h(r+1,t)) + \omega(r, t+1), \tag{2.1}$$

with the initial condition $h(r, 0) = 0$. Here ω is the random variable expressing the height of nucleation and takes a value in $\mathbb{N}$. $\omega(r,t) = 0$ if $t - r$ is even or if $|r| > t$, and

$$w(i,j) = \omega(i-j, i+j-1), \tag{2.2}$$

$(i,j) \in \mathbb{Z}_+^2$ are geometric random variables. The parameter of this random variable is taken to be of the form $a_i b_j$,

$$\mathbb{P}[w(i,j) = k] = (1 - a_i b_j)(a_i b_j)^k, \tag{2.3}$$

for $k \in \mathbb{N}$. In order to consider the effect of external sources at both edges, we take

$$a_j = \begin{cases} \gamma_-, & j = 1, \\ \alpha, & j \geq 2, \end{cases} \tag{2.4}$$

$$b_j = \begin{cases} \gamma_+, & j = 1, \\ \alpha, & j \geq 2. \end{cases} \tag{2.5}$$

The parameter α is related to the frequency of nucleations in the bulk whereas the parameter $\gamma_\pm$ represents the strength of the external sources at the edges. The bigger $\gamma_\pm$ is, the stronger the source is. In the following, we assume $\gamma_- > \gamma_+$ when we do some computations; the results for the case where $\gamma_+ > \gamma_-$ is obtained from the symmetry.

Some snapshots of Monte Carlo simulations are given in Fig.2. When the external sources are not very strong (Fig.(a)), the effects of the external sources are important for some region near the edges whereas the bulk dynamics is important for the curved region at bulk. On the other hand, when the external sources are strong (Fig.(b)) they are dominant for the whole region. In the critical situation (Fig.(c)), they control the whole region but a certain point. More precisely the shape is described as follows. Let us set

$$a(\beta) = \frac{2\alpha}{1-\alpha^2}\left(\alpha + \sqrt{1-\beta^2}\right), \tag{2.6}$$

$$a_{G\pm}(\beta,\gamma) = \frac{\alpha(1 - 2\alpha\gamma + \gamma^2)}{(\gamma-\alpha)(1-\alpha\gamma)} \pm \frac{\alpha(\gamma^2-1)}{(\gamma-\alpha)(1-\alpha\gamma)}\beta, \tag{2.7}$$

and

$$\beta_- = \frac{(1-\alpha^2)(\gamma_-^2-1)}{1+\alpha^2-4\alpha\gamma_- + \gamma_-^2 + \alpha^2\gamma_-^2}, \tag{2.8}$$

$$\beta_+ = -\frac{(1-\alpha^2)(\gamma_+^2-1)}{1+\alpha^2-4\alpha\gamma_+ + \gamma_+^2 + \alpha^2\gamma_+^2}. \tag{2.9}$$

Notice $\beta_- < \beta_+ \Leftrightarrow \gamma_+\gamma_- < 1$. Then the thermodynamic shape is given by the following.
(i) When $\beta_- < \beta_+$,

$$h(r = 2\beta N, t = 2N)/N \sim \begin{cases} a_{G-}(\beta, \gamma_-), & \beta < \beta_-, \\ a(\beta), & \beta_- < \beta < \beta_+, \\ a_{G+}(\beta, \gamma_+), & \beta > \beta_+. \end{cases} \tag{2.10}$$

(ii) When $\beta_- > \beta_+$,

$$h(r = 2\beta N, t = 2N)/N \sim \begin{cases} a_{G-}(\beta, \gamma_-), & \beta < \beta_c, \\ a_{G+}(\beta, \gamma_+), & \beta > \beta_c, \end{cases} \tag{2.11}$$

with β_c being the solution of $a_{G-}(\beta, \gamma_-) = a_{G+}(\beta, \gamma_+)$.

The fluctuation properties of the model change drastically at the connecting points $\beta_\pm$ and β_c of the limiting shapes. Let us define the two scaled height variables. The first one is

$$H_N(\tau, \beta_0) = \frac{h(r = 2\beta_0 N + 2c(\beta_0)N^{\frac{2}{3}}\tau, t = 2N) - a(\beta_0 + \frac{c(\beta_0)\tau}{N^{1/3}})N}{d(\beta_0)N^{\frac{1}{3}}}, \tag{2.12}$$

where

$$d(\beta) = \frac{\alpha^{\frac{1}{3}}}{(1-\alpha^2)(1-\beta^2)^{\frac{1}{6}}}(\sqrt{1+\beta} + \alpha\sqrt{1-\beta})^{\frac{2}{3}}(\sqrt{1-\beta} + \alpha\sqrt{1+\beta})^{\frac{2}{3}}, \tag{2.13}$$

$$c(\beta) = \alpha^{-\frac{1}{3}}(1-\beta^2)^{\frac{2}{3}}(\sqrt{1+\beta} + \alpha\sqrt{1-\beta})^{\frac{1}{3}}(\sqrt{1-\beta} + \alpha\sqrt{1+\beta})^{\frac{1}{3}}. \tag{2.14}$$

The second is

$$H_N^{(G\pm)}(\beta_0, \gamma) = \frac{h(r = 2\beta_0 N, t = 2N) - a_{G\pm}(\beta_0, \gamma)N}{d_G(\gamma)N^{\frac{1}{2}}}, \tag{2.15}$$

where

$$d_G(\gamma) = \frac{\sqrt{\alpha\gamma(1+\alpha^2-4\alpha\gamma+\gamma^2+\alpha^2\gamma^2)}}{(1-\alpha\gamma)(\gamma-\alpha)}. \tag{2.16}$$

About the one point height fluctuation, we have the following.

Theorem 2.1.
(i) When $\beta_- < \beta_+$.
a) For $\beta_- < \beta_0 < \beta_+$,

$$\lim_{N\to\infty} \mathbb{P}[H_N(0,\beta_0) \le s] = F_2(s), \tag{2.17}$$

where F_2 denotes the GUE Tracy-Widom distribution, which is the distribution of the largest eigenvalue in GUE [5].
b) For $\beta_0 = \beta_-$ or $\beta_0 = \beta_+$,

$$\lim_{N\to\infty} \mathbb{P}[H_N(0,\beta_0) \le s] = F_1(s)^2. \tag{2.18}$$

where F_1 is the GOE Tracy-Widom distribution [6]. Thus F_1^2, which is denoted by GOE^2, means the distribution of the larger of the largest eigenvalues in two independent GOEs.
c) For $\beta_0 < \beta_-$ or $\beta_0 > \beta_+$, the fluctuation is Gaussian. One has

$$\lim_{N\to\infty} \mathbb{P}[H_N^{(G\pm)}(\beta_0,\gamma_\pm) \le s] = \frac{1}{\sqrt{2\pi|\beta_\pm - \beta_0|}} \int_{-\infty}^{s} d\xi e^{-\frac{\xi^2}{2\pi|\beta_\pm-\beta_0|}}. \tag{2.19}$$

(ii) When $\beta_- > \beta_+$.
a) For $\beta_0 < \beta_c$ or $\beta_0 > \beta_c$, the fluctuation is Gaussian (cf. (i-c)).
b) For $\beta_0 = \beta_c$, the fluctuation might be given by

$$\lim_{N\to\infty} \mathbb{P}[H_N^{(G_-)}(\beta_0,\gamma_-) \le s] = \int_{-\infty}^{s} d\xi_1 \frac{e^{-\frac{\xi_1^2}{2(\beta_- - \beta_0)}}}{\sqrt{2\pi(\beta_- - \beta_0)}} \int_{-\infty}^{s} d\xi_2 \frac{e^{-\frac{\xi_2^2}{2(\beta_0-\beta_+)}\frac{d_G(\gamma_-)^2}{d_G(\gamma_+)^2}}}{\sqrt{2\pi(\beta_0 - \beta_+)\frac{d_G(\gamma_+)}{d_G(\gamma_-)}}}. \tag{2.20}$$

(iii) When $\beta_- = \beta_+$.
a) For $\beta_0 < \beta_-$ or $\beta_0 > \beta_-$, the fluctuation is Gaussian (cf. (i-c)).
b) For $\beta_0 = \beta_-$,

$$\lim_{N\to\infty} \mathbb{P}[H_N(0,\beta_0) \le s] = F_0(s). \tag{2.21}$$

Here F_0 is a certain distribution with mean zero explained in [17, 18, 27].

These are obtained as corollaries of the theorem in the next section. The special case of the height at the origin ($\beta_0 = 0$) was previously studied in section 4 of [27] using the connection of the problem to the combinatorics of Young tableaux. The limiting distribution was obtained using the Riemann-Hilbert method and the results were given in terms of the solution to the Painlevé equation. In [19], for the continuous model, the basic picture of this theorem was expected based on a physical argument but has not been shown explicitly. These distributions also appear in the fluctuation properties of the one-dimensional asymmetric simple exclusion process (ASEP) [9, 10].

3 Multi-point Height Fluctuation

As was observed in [21,22,25,28], the equal time multi-point correlation of the PNG model can be analyzed by extending the original model to the multi-layer version. The weight of the multi-layer version is equivalent to that of non-intersecting many-body random walk. In particular, for the model under consideration, this can be borrowed from the results in [22]. Following [22], we only consider an odd time $M = 2N - 1$ in the sequel. Let us consider the weight for the configuration $\{x_j^r\} \equiv \bar{x}$ of n non-intersecting paths from the time $r = -M$ to $r = M$ given by

$$w_{n,M}(\bar{x}) = \prod_{r=-M}^{M-1} \det(\phi_{r,r+1}(x_i^r, x_j^{r+1}))_{i,j=1}^n, \tag{3.1}$$

where

$$\phi_{2j-1,2j}(x,y) = \begin{cases} (1-a_{j+N})a_{j+N}^{y-x}, & y \geq x, \\ 0, & y < x, \end{cases} \tag{3.2}$$

$$\phi_{2j,2j+1}(x,y) = \begin{cases} 0, & y > x, \\ (1-b_{N-j})b_{N-j}^{x-y}, & y \leq x, \end{cases} \tag{3.3}$$

and $x_i^M = x_i^{-M} = 1 - i$ $(i = 1, 2, \cdots, n)$ is fixed. Note that the same weight gives a weight for a time evolution by the time M of the PNG model. For the PNG model with external sources we consider in this paper, the parameters a_j, b_j's are taken to be (2.4), (2.5). Strictly speaking, the weight of the multi-layer PNG model and the weight (3.1) with (2.4),(2.5) and $n = N$ are slightly different. Essentially the same remark was already given in [25]. The difference is, however, negligible in the scaling limit in which we are mainly interested in this paper.

For each fixed $\gamma_\pm$, as $N \to \infty$, we have the following results.

Theorem 3.1.
(i) When $\beta_- < \beta_+$.
a)For $\beta_- < \beta_0 < \beta_+$, the equal time multi-point distribution function is described by the following Fredholm determinant.

$$\begin{aligned} &\lim_{N\to\infty} \mathbb{P}[H_N(\tau_1, \beta_0) \leq s_1, \cdots, H_N(\tau_m, \beta_0) \leq s_m] \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1=1}^{m} \int d\xi_1 \cdots \sum_{n_k=1}^{m} \int d\xi_k \; \mathcal{G}(\tau_{n_1}, \xi_1) \cdots \mathcal{G}(\tau_{n_k}, \xi_k) \det(\mathcal{K}(\tau_{n_l}, \xi_l; \tau_{n_{l'}}, \xi_{l'}))_{l,l'=1}^k, \\ &\equiv \det(1 + \mathcal{K}\mathcal{G}), \end{aligned} \tag{3.4}$$

where $\mathcal{G}(\tau_j, \xi) = -\chi_{(s_j,\infty)}(\xi)$ $(j = 1, 2, \cdots, m)$.(χ_J is the characteristic function.) The kernel $\mathcal{K}$ is the extended Airy kernel,

$$\mathcal{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) = \begin{cases} \int_0^\infty d\lambda e^{-\lambda(\tau_1-\tau_2)} \mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda), & \tau_1 \geq \tau_2, \\ -\int_{-\infty}^0 d\lambda e^{-\lambda(\tau_1-\tau_2)} \mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda), & \tau_1 < \tau_2. \end{cases} \tag{3.5}$$

b) For $\beta_0 = \beta_-$ or $\beta_0 = \beta_+$, we have (3.4) with a different kernel. It is denoted as $\mathcal{K}_{12}$ and is given by

$$\begin{aligned}
&\mathcal{K}_{12}(\tau_1, \xi_1; \tau_2, \xi_2) \\
&= \begin{cases} \mathcal{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) + \mathrm{Ai}(\xi_1) \int_0^\infty d\lambda e^{-\tau_2 \lambda} \mathrm{Ai}(\xi_2 - \lambda), & \tau_2 > 0, \\ \mathcal{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) - \mathrm{Ai}(\xi_1) \int_0^\infty d\lambda e^{\tau_2 \lambda} \mathrm{Ai}(\xi_2 + \lambda) + \mathrm{Ai}(\xi_1) e^{\frac{\tau_2^3}{3} - \xi_2 \tau_2}, & \tau_2 < 0. \end{cases}
\end{aligned} \tag{3.6}$$

c) In the region where $\beta_0 < \beta_-$ or $\beta_0 > \beta_+$, the fluctuation is equivalent to those of the Brownian motion. In terms of the Fredholm representation, when $\beta_1 < \beta_2 < \cdots < \beta_m < \beta_-$, we have

$$\lim_{N \to \infty} \mathbb{P}[H_N^{(G-)}(\beta_1, \gamma_-) \le s_1, \cdots, H_N^{(G-)}(\beta_m, \gamma_-) \le s_m] = \det(1 + \mathcal{K}\mathcal{G}), \tag{3.7}$$

where the kernel is

$$\mathcal{K}_{G-}(\beta_1, \xi_1; \beta_2, \xi_2) = \begin{cases} \dfrac{e^{-\frac{\xi_1^2}{2(\beta_- - \beta_1)}}}{\sqrt{2\pi(\beta_- - \beta_1)}}, & \beta_1 \ge \beta_2, \\ \dfrac{e^{-\frac{\xi_1^2}{2(\beta_- - \beta_1)}}}{\sqrt{2\pi(\beta_- - \beta_1)}} - \dfrac{e^{-\frac{(\xi_2 - \xi_1)^2}{2(\beta_2 - \beta_1)}}}{\sqrt{2\pi(\beta_2 - \beta_1)}}, & \beta_1 < \beta_2. \end{cases} \tag{3.8}$$

The results for the case where $\beta_+ < \beta_1 < \beta_2 < \cdots < \beta_m$ is analogous.
(ii) When $\beta_- > \beta_+$.
a)For $\beta < \beta_c$, the same fluctuation as (3.8) *is obtained.*
(iii) When $\beta_- = \beta_+$.
a)For $\beta < \beta_-$ and $\beta > \beta_-$, the same fluctuation as (3.8) *is obtained.*
b) For $\beta = \beta_c$, the results can be obtained as a limiting case of Theorem 5.1 in section 5.

Remarks.

1. The process characterized by the Fredholm determinant with the extended Airy kernel [30,31] in (i-a) is called the Airy process [21,22]. This is the same as the process of the largest eigenvalue in the Dyson's Brownian Motion model [29] between unitary ensembles.

2. For (i-c) we can easily calculate the multi-point joint distributions. For instance the

two-point joint distribution is calculated as

$$
\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}[H_N^{(G-)}(\beta_1,\gamma_-) \le s_1, H_N^{(G-)}(\beta_2,\gamma_-) \le s_2] \\
&= 1 - \int_{s_1}^{\infty} d\xi_1 \mathcal{K}_{G_-}(\beta_1,\xi_1;\beta_1,\xi_1) - \int_{s_2}^{\infty} d\xi_2 \mathcal{K}_{G_-}(\beta_2,\xi_2;\beta_2,\xi_2) \\
&\quad + \frac{1}{2}\int_{s_1}^{\infty} d\xi_1 \int_{s_2}^{\infty} d\xi_2 \begin{vmatrix} \mathcal{K}_{G_-}(\beta_1,\xi_1;\beta_1,\xi_1) & \mathcal{K}_{G_-}(\beta_1,\xi_1;\beta_2,\xi_2) \\ \mathcal{K}_{G_-}(\beta_2,\xi_2;\beta_1,\xi_1) & \mathcal{K}_{G_-}(\beta_2,\xi_2;\beta_2,\xi_2) \end{vmatrix} \\
&\quad + \frac{1}{2}\int_{s_1}^{\infty} d\xi_1 \int_{s_2}^{\infty} d\xi_2 \begin{vmatrix} \mathcal{K}_{G_-}(\beta_2,\xi_2;\beta_2,\xi_2) & \mathcal{K}_{G_-}(\beta_2,\xi_2;\beta_1,\xi_1) \\ \mathcal{K}_{G_-}(\beta_1,\xi_1;\beta_2,\xi_2) & \mathcal{K}_{G_-}(\beta_1,\xi_1;\beta_1,\xi_1) \end{vmatrix} \\
&= \int_{-\infty}^{s_1} d\xi_1 \int_{-\infty}^{s_2} d\xi_2 \frac{e^{\frac{-(\xi_2-\xi_1)^2}{2(\beta_2-\beta_1)}}}{\sqrt{2\pi(\beta_2-\beta_1)}} \frac{e^{\frac{-\xi_2^2}{2(\beta_- - \beta_2)}}}{\sqrt{2\pi(\beta_- - \beta_2)}}. \qquad (3.9)
\end{aligned}
$$

Note that the 3 by 3 determinant, $\det[K_{G_-}(t_i,\xi_i;t_j,\xi_j)]_{i,j=1}^3$ becomes zero when $t_j = \beta_1, \beta_2$ since the 2 by 2 determinant at each β_j, $\det[K_{G_-}(\beta_k,\xi_i;\beta_k,\xi_j)]_{i,j=1}^3$ $(k=1,2)$, vanishes. Hence bigger determinants also vanish. The integrand in (3.9) represents the propagation of a Brownian particle.

Proof. Here we prove (i-a), (i-c), (ii-a) and (iii-a). The remainings, (i-b) and (iii-b), will be obtained as corollaries of the results in the following sections.

First we start our proof with the fact that equal time multi-point correlation of height fluctuations at the odd time $M = 2N-1$ has the Fredholm representation [22],

$$
\begin{aligned}
&\mathbb{P}\left[h(r_1,M) \le l_1, h(r_2,M), \le l_2 \cdots, h(r_m,M) \le l_m\right] \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1=1}^{m} \sum_{x_1} \cdots \sum_{i_k=1}^{m} \sum_{x_k} g(r_{i_1},x_1)\cdots g(r_{i_k},x_k) \det(K_N(r_{i_l},x_l;r_{i_{l'}},x_{l'}))_{l,l'=1}^k \\
&\equiv \det(1+K_N g), \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
&g(r_j,x_i) = -\chi_{(l_j,\infty)}(x_i), \\
&K_N(r_1=2u_1,x_1;r_2=2u_2,x_2) = \tilde{K}_N(2u_1,x_1;2u_2,x_2) - \phi_{2u_1,2u_2}(x_1,x_2), \qquad (3.11) \\
&\tilde{K}_N(2u_1,x_1;2u_2,x_2), \\
&= \frac{(1-\alpha)^{2(u_2-u_1)}}{(2\pi i)^2} \int_{C_{R_1}} \frac{dz_1}{z_1} \int_{C_{R_2}} \frac{dz_2}{z_2} \frac{z_2^{x_2}}{z_1^{x_1}} \frac{z_1}{z_1-z_2} \frac{(1-\alpha/z_1)^{N-1+u_1}(1-\alpha z_2)^{N-1-u_2}}{(1-\alpha z_1)^{N-1-u_1}(1-\alpha/z_2)^{N-1+u_2}} \\
&\quad \times \frac{1-\gamma_-/z_1}{1-\gamma_-/z_2} \frac{1-\gamma_+ z_2}{1-\gamma_+ z_1}, \qquad (3.12) \\
&\phi_{r_1,r_2}(x_1,x_2) = \begin{cases} \frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i} \int_{C_1} \frac{dz}{z} z^{x_2-x_1}[(1-\alpha z)(1-\alpha/z)]^{u_2-u_1}, & u_2 > u_1, \\ 0, & u_2 \leqslant u_1. \end{cases} \qquad (3.13)
\end{aligned}
$$

In (3.12), C_{R_i} means a contour with a radius R_i and C_1 is the unit circle. In both cases, they enclose the origin anticlockwise. One takes the radiuses C_{R_1}, C_{R_2} in a way that $\gamma_- < C_{R_2} < C_{R_1} < 1/\gamma_+$. Note that (3.10) is valid for $\gamma_+\gamma_- < 1$.

Next we discuss the asymptotics by applying the saddle point method to the kernel (3.12) and (3.13).

proof of (i-a)

We prove (3.4) and (3.5) along the same line as the derivation of the Proposition 4.1 in [25]. We give only the outline of the proof.

• **Asymptotics of** $\tilde{K}_N$

We set

$$\begin{aligned} &\tilde{K}_N(r_1 = 2u_1, x_1; r_2 = 2u_2, x_2) \\ &= \frac{(1-\alpha)^{2(u_2-u_1)}}{(2\pi i)^2} \int_{C_{R_1}} \frac{dz_1}{z_1} \int_{C_{R_2}} \frac{dz_2}{z_2} \frac{z_2^{x_2-N(\mu_2-1)}}{z_1^{x_1-N(\mu_1-1)}} e^{N(g_{\mu_1,\beta_1}(z_1)+g_{\mu_2,\beta_2}(1/z_2))} \\ &\times \frac{z_1}{z_1-z_2} \frac{1-\gamma_-/z_1}{1-\gamma_-/z_2} \frac{1-\gamma_+ z_2}{1-\gamma_+ z_1}, \end{aligned} \tag{3.14}$$

where $\beta_1 = u_1/N, \beta_2 = -u_2/N$, μ_1, μ_2 are arbitrary constants at this stage and

$$g_{\mu,\beta}(z) = (1+\beta)\log(z-\alpha) - (1-\beta)\log(1-\alpha z) - (\mu+\beta)\log z. \tag{3.15}$$

Using $a(\beta), c(\beta), d(\beta)$ defined in (2.6), (2.13) and (2.14) respectively, we consider the scaling,

$$\beta_i = \beta_0 + \frac{c(\beta_0)}{N^{\frac{1}{3}}}\tau_i, \tag{3.16}$$

$$x_i = a(\beta_i)N + d(\beta_0)N^{\frac{1}{3}}\xi_i. \tag{3.17}$$

When we fix $\mu_c(\beta) = a(\beta)+1$, two critical points of $g_{\mu,\beta}(z)$ are combined to the double critical point $p_c(\beta)$ given by

$$p_c(\beta) = p(\mu_c(\beta), \beta) = \frac{\sqrt{1+\beta}+\alpha\sqrt{1-\beta}}{\sqrt{1-\beta}+\alpha\sqrt{1+\beta}}, \tag{3.18}$$

where $g'_{\mu_c(\beta),\beta}(p_c(\beta)) = g''_{\mu_c(\beta),\beta}(p_c(\beta)) = 0$. Since $\gamma_- < p_c(\beta) < \frac{1}{\gamma_+}$ for $\alpha < \gamma < \frac{1}{\alpha}$, $\beta_- < \beta < \beta_+$, we can deform the contour of z_i to

$$z_1 = p_c(\beta_1)\left(1 - \frac{i}{d(\beta_0)N^{1/3}}w_1\right) \sim p_c(\beta_0)\left(1 + \frac{1}{d(\beta_0)N^{1/3}}(\tau_1 - iw_1)\right), \tag{3.19}$$

$$\frac{1}{z_2} = p_c(\beta_2)\left(1 - \frac{i}{d(\beta_0)N^{1/3}}w_2\right) \sim \frac{1}{p_c(\beta_0)}\left(1 - \frac{1}{d(\beta_0)N^{1/3}}(\tau_2 + iw_2)\right). \tag{3.20}$$

Then we find

$$N g_{\mu_c(\beta_1),\beta_1}(z_1) \sim N\lambda_0(\beta_0) + \lambda_1(\beta_0)c(\beta_0)N^{2/3}\tau_1 + \lambda_2(\beta_0)c(\beta_0)^2N^{1/3}\tau_1^2 + \lambda_3(\beta_0)c(\beta_0)^3\tau_1^3 + \frac{i}{3}w_1^3, \tag{3.21}$$

$$N g_{\mu_c(\beta_2),\beta_2}(\frac{1}{z_2}) \sim -N\lambda_0(\beta_0) - \lambda_1(\beta_0)c(\beta_0)N^{2/3}\tau_2 - \lambda_2(\beta_0)c(\beta_0)^2N^{1/3}\tau_2^2 - \lambda_3(\beta_0)c(\beta_0)^3\tau_2^3 + \frac{i}{3}w_2^3, \tag{3.22}$$

where

$$\lambda_i(\beta_0) = \frac{d^i}{d\beta^i} g_{\mu_c(\beta),\beta}(p_c(\beta))\bigg|_{\beta=\beta_0}. \tag{3.23}$$

Similarly one gets

$$\frac{z_1}{z_1 - z_2} \sim -\frac{d(\beta_0)N^{1/3}}{\tau_2 - \tau_1 + i(w_1 + w_2)}, \tag{3.24}$$

$$\frac{z_2^{x_2+N(1-\mu_c(\beta_2))}}{z_1^{x_1+N(1-\mu_c(\beta_1))}} \sim (p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} e^{\xi_2\tau_2-\xi_1\tau_1+i\xi_1w_1+i\xi_2w_2}. \tag{3.25}$$

Substituting (3.19)– (3.25) into (3.14), one obtains

$$\begin{aligned}\tilde{K}_N \sim{}& (1-\alpha)^{2(u_2-u_1)}(p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}}\frac{1}{d(\beta_0)N^{1/3}} \\ & e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1} \\ & \frac{1}{4\pi^2}\int_{\mathrm{Im}w_1=\eta_1} dw_1 \int_{\mathrm{Im}w_2=\eta_2} dw_2 \left(-\frac{1}{\tau_2-\tau_1+i(w_1+w_2)}\right) e^{i\xi_1w_1+i\xi_2w_2+\frac{i}{3}(w_1^3+w_2^3)}\end{aligned} \tag{3.26}$$

$$\begin{aligned}={}& (1-\alpha)^{2(u_2-u_1)}(p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}}\frac{1}{d(\beta_0)N^{1/3}} \\ & e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1} \\ & \int_0^\infty d\lambda e^{-\lambda(\tau_1-\tau_2)}\mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda),\end{aligned} \tag{3.27}$$

where $\eta_i > 0$ is a convergence factor.

- **Asymptotics of** $\phi_{2u_1,2u_2}(x_1,x_2)$

Next we consider the asymptotics of $\phi_{2u_1,2u_2}(x_1,x_2)$. For $u_1 < u_2$

$$\begin{aligned}&\phi_{r_1,r_2}(x_1,x_2) \\ &= \frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i}\int_{C_1}\frac{dz}{z}z^{x_2-x_1}[(1-\alpha z)(1-\alpha/z)]^{u_2-u_1} \\ &= \frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i}\int_{C_1}\frac{dz}{z}z^{x_2-N(\mu_c(\beta_2)-1)-x_1+N(\mu_c(\beta_1)-1)}e^{Ng_{\mu_c(\beta_1),\beta_1}(z)+Ng_{\mu_c(\beta_2),\beta_2}(1/z)}.\end{aligned} \tag{3.28}$$

We set

$$z = p_c(\beta_0)\left(1 + \frac{i\sigma}{d(\beta_0)N^{1/3}}\right) \sim p_c(\beta_1)\left(1 - \frac{1}{d(\beta_0)N^{1/3}}(\tau_1 - i\sigma)\right) \tag{3.29}$$

$$\sim \frac{1}{p_c(\beta_2)}\left(1 - \frac{1}{d(\beta_0)N^{1/3}}(\tau_2 - i\sigma)\right). \tag{3.30}$$

Applying these to $g_{\mu_c(\beta_1),\beta_1}(z)$ and $g_{\mu_c(\beta_2),\beta_2}(1/z)$ respectively, we get

$$Ng_{\mu_c(\beta_1),\beta_1}(z) \sim Ng_{\mu_c(\beta_1),\beta_1}\big(p_c(\beta_1)\big) - \frac{1}{3}(\tau_1 - i\sigma)^3, \tag{3.31}$$

$$Ng_{\mu_c(\beta_2),\beta_2}\left(\frac{1}{z}\right) \sim Ng_{\mu_c(\beta_2),\beta_2}\big(p_c(\beta_2)\big) + \frac{1}{3}(\tau_2 - i\sigma)^3. \tag{3.32}$$

One also obtains

$$z^{x_2 - N(\mu_c(\beta_2)-1) - x_1 + N(\mu_c(\beta_1)-1)} \sim \big(p_c(\beta_0)\big)^{d(\beta_0)N^{1/3}(\xi_2-\xi_1)} e^{i\sigma(\xi_2-\xi_1)}. \tag{3.33}$$

Hence one finds

$$\begin{aligned}
\phi_{r_1,r_2}(x_1,x_2) &\sim (1-\alpha)^{2(u_2-u_1)}\big(p_c(\beta_0)\big)^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} \frac{1}{d(\beta_0)N^{1/3}} \\
&\quad e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)-\frac{\tau_1^3}{3}+\frac{\tau_2^3}{3}} \\
&\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} d\sigma e^{i(\xi_2-\xi_1+\tau_1^2-\tau_2^2)\sigma-(\tau_2-\tau_1)\sigma^2} \qquad (3.34)\\
&= (1-\alpha)^{2(u_2-u_1)}\big(p_c(\beta_0)\big)^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} \frac{1}{d(\beta_0)N^{1/3}} \\
&\quad e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)-\frac{\tau_1^3}{3}+\frac{\tau_2^3}{3}} \\
&\quad \int_{-\infty}^{\infty} d\lambda e^{-\lambda(\tau_1-\tau_2)}\mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda). \qquad (3.35)
\end{aligned}$$

From (3.27) and (3.35) we get the desired expression. Note that the pre-factor of the extended Airy kernel does not affect the Fredholm determinant.

proof of (i-c),(ii-a) and (iii-a)

One can prove (i-c) in a fairly similar manner to (i-a) but an essential modification is necessary for the proof of the other two cases. This results from the fact that the model defined by (2.1)– (2.5) is not well-defined in the latter case since the parameter of geometric distribution at $i = 1, j = 1$, which is equal to $\gamma_+\gamma_-$, is greater than 1. Thus we consider a slightly modified model in which the parameter of the random variable at the point (1,1) is zero as in [27]. When $\gamma_+\gamma_- < 1$, the modification is unnecessary but we consider the modified model here because the modification does not change the asymptotic properties of the model and allows us to treat all cases in a parallel fashion.

For $\gamma_+\gamma_- < 1$ we can get easily the relation of the equal time multi-point correlation function between these models by generalizing the relation in one-point case [27]. Let $h^+(r_i,t)$ represent the height in the original model and $h(r_i,t)$ the height of the modified model. One has

$$\begin{aligned}&\mathbb{P}[h(r_1,t=M\equiv 2N-1)<l_1,\cdots,h(r_m,t=M)<l_m]\\&=\frac{\mathbb{P}[h^+(r_1,M)<l_1,\cdots,h^+(r_m,M)<l_m]-\gamma_+\gamma_-\mathbb{P}[h^+(r_1,M)<l_1-1,\cdots,h^+(r_m,M)<l_m-1]}{1-\gamma_+\gamma_-}.\end{aligned} \tag{3.36}$$

When $\gamma_+\gamma_- < 1$, $\mathbb{P}[h^+(r_1,M)<l_1,\cdots,h^+(r_m,M)<l_m]$ is represented as the Fredholm determinant (3.10) with the kernel (3.11)– (3.13). The problem is that the Fredholm determinant is not well-defined for $\gamma_+\gamma_- > 1$ (See (3.45) below). We would like to obtain another representation applicable for the case where $\gamma_+\gamma_- > 1$ by modifying the Fredholm representation.

We start from the Fredholm representation (3.10) for $\gamma_+\gamma_- < 1$. Using the relation

$$\frac{z_1}{z_1-z_2}\frac{1-\gamma_+z_2}{1-\gamma_+z_1}\frac{1-\gamma_-/z_1}{1-\gamma_-/z_2}=\frac{z_2}{z_1-z_2}+\frac{1-\gamma_+\gamma_-}{(1-\gamma_+z_1)(1-\gamma_-/z_2)}, \tag{3.37}$$

the kernel $\tilde{K}_N$ can be divided into two terms,

$$\begin{aligned}&\tilde{K}_N(2u_1,x_1;2u_2,x_2)\\&=\frac{1}{(2\pi i)^2}\int_{C_{R_1}}\frac{dz_1}{z_1}\int_{C_{R_2}}\frac{dz_2}{z_2}\frac{z_2^{x_2}}{z_1^{x_1}}\frac{z_2}{z_1-z_2}\frac{F(u_1,z_1)}{F(u_2,z_2)}\\&+(1-\gamma_+\gamma_-)\frac{1}{2\pi i}\int_{C_{R_1}}\frac{dz_1}{z_1}\frac{1}{z_1^{x_1}}F(u_1,z_1)\frac{1}{1-\gamma_+z_1}\times\frac{1}{2\pi i}\int_{C_{R_2}}\frac{dz_2}{z_2}\frac{z_2^{x_2}}{F(u_2,z_2)}\frac{1}{1-\gamma_-/z_2}\\&\equiv\tilde{K}_{2N}(u_1,x_1;u_2,x_2)+(1-\gamma_+\gamma_-)D_+(u_1,x_1)D_-(u_2,x_2),\end{aligned} \tag{3.38}$$

where

$$F(u,z)=\frac{(1-\alpha/z)^{N-1+u}}{(1-\alpha)^{2u}(1-\alpha z)^{N-1-u}}. \tag{3.39}$$

Remember that this expression was derived under the condition $\alpha<\gamma_-<R_2<R_1<1/\gamma_+<1/\alpha$. The Fredholm determinant can be deformed to

$$\begin{aligned}&\det[1+K_Ng]\\&=\det[1+K_{2N}g]\left\{1-(1-\gamma_+\gamma_-)\sum_{k,k'=1}^{m}\sum_{x_1=l_k}^{\infty}\sum_{x_2=l_{k'}}^{\infty}D_-(u_k,x_2)E(u_k,x_1;u_{k'},x_2)D_+(u_{k'},x_2)\right\},\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}&K_{2N}(u_1,x_1;u_2,x_2)=\tilde{K}_{2N}(u_1,x_1;u_2,x_2)-\phi_{2u_1,2u_2}(x_1,x_2),\\&E(u_1,x_1;u_2,x_2)=(1-K_{2N})^{-1}(u_1,x_1;u_2,x_2).\end{aligned} \tag{3.41}$$

In the above equation let us focus our attention to the contribution of terms,

$$(1-\gamma_+\gamma_-)\sum_{k=1}^{m}\sum_{x=l_k}^{\infty} D_-(u_k,x)D_+(u_k,x), \tag{3.42}$$

which arises from the 'delta function' part of $E(u_1,x_1;u_2,y_1)$. Here one divides $D_\pm$ into two parts,

$$\begin{aligned} D_-(u,x) &= \frac{1}{2\pi i}\int_{C_{R_2'}}\frac{dz_2}{z_2}\frac{z_2^x}{F(u,z_2)}\frac{1}{1-\gamma_-/z_2}+\frac{\gamma_-^x}{F(u,\gamma_-)} \equiv D_{1-}(u,x)+D_{2-}(u,x), \\ D_+(u,x) &= \frac{1}{2\pi i}\int_{C_{R_1'}}\frac{dz_1}{z_1}\frac{1}{z_1^x}\frac{F(u,z_1)}{1-\gamma_+z_1}+\gamma_+^xF(u,1/\gamma_+) \equiv D_{1+}(u,x)+D_{2+}(u,x), \end{aligned} \tag{3.43}$$

where the radiuses of contours R_i' are taken to satisfy $\alpha < R_2' < \gamma_- < 1/\gamma_+ < R_1' < 1/\alpha$. Notice the second terms appear from the contribution of the poles at γ_- in $D_-E(u,x)$ and at $1/\gamma_+$ in $D_+(k,x)$. One finds

$$\begin{aligned} &(1-\gamma_+\gamma_-)\sum_{x=l_k}^{\infty} D_+(u_k,x)D_-(u_k,x) \\ &= (1-\gamma_+\gamma_-)\sum_{k=1}^{m}\sum_{x=l_k}\Big[D_{1+}(u_k,x)D_{1-}(u_k,x)+D_{1+}(u_k,x)D_{2-}(u_k,x) \\ &\quad + D_{2+}(u_k,x)D_{1-}(u_k,x)+D_{2+}(u_k,x)D_{2-}(u_k,x)\Big]. \end{aligned} \tag{3.44}$$

The last term can be rewritten as

$$(1-\gamma_+\gamma_-)\sum_{x=l_k}^{\infty} D_{2+}(u_k,x)D_{2-}(u_k,x) = (1-\gamma_+\gamma_-)\sum_{x=l_k}^{\infty}(\gamma_+\gamma_-)^x\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)} = (\gamma_+\gamma_-)^{l_k}\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}. \tag{3.45}$$

One notices that the series on the middle is divergent when $\gamma_+\gamma_- > 1$ but that the difficulty is avoided in the right most expression. In addition one can easily find that such a difficulty does not arise for the other terms in (3.44) and the remaining terms of $(1-\gamma_+\gamma_-)\sum D_-(u_k,x_1)E(u_k,x_1;u_{k'},x_2)D_+(u_{k'},x_2)$ in (3.40) with the contribution of (3.42) subtracted,

$$\sum_{k,k'=1}^{m}\sum_{x_1=l_k}^{\infty}\sum_{x_2=l_{k'}}^{\infty} D_-(u_k,x_1)E'(u_k,x_1;u_{k'},x_2)D_+(u_{k'},x_2), \tag{3.46}$$

where

$$E'(u_k,x_1;u_{k'},x_2) = \begin{cases} E(u_k,x_1;u_{k'},x_2)-\delta_k^+(x_1,x_2), & \text{for } k=k', \\ E(u_k,x_1;u_{k'},x_2), & \text{for } k\neq k', \end{cases} \tag{3.47}$$

and

$$\sum_{y=l_k}^{\infty} \delta_k^+(x,y) f(y) = f(x). \tag{3.48}$$

Eventually one obtains the representation of $\mathbb{P}[h^+(r_1, M) < l_1, \cdots, h^+(r_m, M) < l_m]$,

$$\begin{aligned}
&\mathbb{P}[h^+(r_1, M) < l_1, \cdots, h^+(r_m, M) < l_m] \\
&= \det\left[1 + K_{2N} g\right] \Bigg(1 - \sum_{k=1}^{m} (\gamma_+\gamma_-)^{l_k} \frac{F(u_k, 1/\gamma_+)}{F(u_k, \gamma_-)} \\
&\quad - (1 - \gamma_+\gamma_-) \Bigg\{ \sum_{k,k'=1}^{m} \sum_{x_1=l_k}^{\infty} \sum_{x_2=l_{k'}}^{\infty} D_-(u_k, x_1) E'(u_k, x_1; u_{k'}, x_2) D_+(u_{k'}, x_2) \\
&\quad + \sum_{k=1}^{m} \sum_{x=l_k} \Big[D_{1+}(u_k, x) D_{1-}(u_k, x) + D_{1+}(u_k, x) D_{2-}(u_k, x) + D_{2+}(u_k, x) D_{1-}(u_k, x) \Big] \Bigg\} \Bigg),
\end{aligned} \tag{3.49}$$

which is well-defined for $\gamma_+\gamma_- > 1$ as well. Note that, when $\gamma_+\gamma_- > 1$, the original meaning as a probability is lost in (3.49).

Next we consider the asymptotics of (3.49). We set $u_i = \beta_i N, l_i = a_{G_-}(\beta_i, \gamma_-)N + d_G(\gamma_-)N^{\frac{1}{2}} s_i$ $\left(x_i = a_{G_-}(\beta_i, \gamma_-)N + d_G(\gamma_-)N^{\frac{1}{2}} \xi_i\right)$ and $N \to \infty$. Let us notice

$$D_{1+}(u, x) = \frac{(1-\alpha)^{-2u}}{2\pi i} \int_{C_{R_1'}} \frac{dz}{z} \frac{1 - \alpha z}{(1 - \alpha/z)(1 - \gamma_+ z)} e^{N g_{y+1,\beta}(z)}, \tag{3.50}$$

where $u = N\beta$ and $x = Ny$. Since

$$y = a_{G_-}(\beta, \gamma_-) + d_G(\gamma_-) \frac{\xi}{N^{\frac{1}{2}}} \equiv y_0 + \delta y, \tag{3.51}$$

the saddle point p_c^G for $g_{y+1,\beta}(z)$ is known to be

$$p_c^G \sim \gamma_- \left(1 + \frac{\xi}{(\beta_- - \beta) d_G N^{\frac{1}{2}}} \right). \tag{3.52}$$

Changing the path of z in a way that it crosses p_c^G,

$$z = p_c^G \left(1 + \frac{iw}{(\beta_- - \beta) d_G N^{\frac{1}{2}}} \right) = \gamma_- \left(1 + \frac{iw + \xi}{(\beta_- - \beta) d_G N^{\frac{1}{2}}} \right), \tag{3.53}$$

we get

$$
\begin{aligned}
&g_{y+1,\beta}(z) \\
&= g_{y+1,\beta}\big(p_c^G(y,\beta)\big) + \frac{1}{2!} g''_{y+1,\beta}\big(p_c^G(y,\beta)\big)\left\{z - p_c^G(y,\beta)\right\}^2 \\
&= g_{y_0+1,\beta}\big(p_c^G(y_0,\beta)\big) - \frac{d_G \xi \log \gamma_-}{N^{\frac{1}{2}}} - \frac{\xi^2 + w^2}{2(\beta_0 - \beta)N}.
\end{aligned} \tag{3.54}
$$

In the second equality we used

$$
\begin{aligned}
g_{y+1,\beta}\big(p_c^G(y,\beta)\big) &\sim g_{y_0+1,\beta}(p_c^G(y_0,\beta)) + \left.\frac{\partial g_{y+1,\beta}}{\partial y}\right|_{y=y_0} \delta y + \frac{1}{2!} \left.\frac{\partial^2 g_{y+1,\beta}}{\partial^2 y}\right|_{y=y_0} \delta y^2 \\
&= g_{y_0+1,\beta}(p_c^G(y_0,\beta)) - \frac{d_G \xi \log \gamma_-}{N^{\frac{1}{2}}} - \frac{\xi^2}{2(\beta_0 - \beta)N},
\end{aligned} \tag{3.55}
$$

and

$$
\frac{1}{2!} g''_{y+1,\beta}\big(p_c^G(y,\beta)\big)\left\{z - p_c^G(y,\beta)\right\}^2 \sim \frac{1}{2!} g''_{y_0+1,\beta}\big(p_c^G(y_0,\beta)\big)\left\{z - p_c^G(y,\beta)\right\}^2 = -\frac{w^2}{2(\beta_0 - \beta)N}. \tag{3.56}
$$

Combining the above, one finally finds

$$
D_{1+}(u,x) \sim \mathcal{D}_{1+}(\beta,\xi) \equiv \frac{F(u,\gamma_-)}{(1-\gamma_+\gamma_-)\gamma_-^x} \frac{e^{-\frac{\xi^2}{2(\beta_- - \beta)}}}{\sqrt{2\pi(\beta_- - \beta)} d_G(\gamma_-) N^{\frac{1}{2}}}. \tag{3.57}
$$

Similarly, one can show

$$
\begin{aligned}
D_{1-}(u,x) \sim \mathcal{D}_{1-}(\beta,\xi) &\equiv \frac{1}{F(u,1/\gamma_+)(1-\gamma_+\gamma_-)\gamma_+^x} \frac{e^{-\frac{\xi^2}{2(\beta-\beta_+)}\frac{d_G(\gamma_-)^2}{d_G(\gamma_+)^2}}}{\sqrt{2\pi(\beta - \beta_+)}\frac{d_G(\gamma_+)}{d_G(\gamma_-)} d_G(\gamma_-) N^{\frac{1}{2}}} \\
&\quad \times e^{-\frac{a_{G-}(\gamma_-) - a_{G+}(\gamma_+)}{(\beta-\beta_+)} \times \frac{d_G(\gamma_+)}{d_G(\gamma_-)} \xi N^{\frac{1}{2}}} e^{-\frac{(a_G(\gamma_-) - a_G(\gamma_+))^2}{4(\beta-\beta_+)} \frac{d_G(\gamma_+)}{d_G(\gamma_-)} \times N}.
\end{aligned} \tag{3.58}
$$

From these relations the asymptotics of $\tilde{K}_{2N}$ is also obtained easily,

$$
\tilde{K}_{2N}(u_1,x_1;u_2,x_2) \sim (1-\gamma_+\gamma_-)\mathcal{D}_{1+}(\beta_1,\xi_1)\mathcal{D}_{1-}(\beta_2,\xi_2). \tag{3.59}
$$

In fact one can find that (3.59) does not contribute to the Fredholm determinant by the following discussion. First we consider the asymptotics when $\beta_+ = \beta_- = \beta_0$ deforming the contour of z_i such that they cross the double critical point $p_c(\beta_0)$,

$$
\tilde{K}_{2N}(\beta_0 N, x_1; \beta_0 N, x_2) = \frac{1}{(2\pi i)^2} \int_{C_{R_1}} \frac{dz_1}{z_1} \int_{C_{R_2}} \frac{dz_2}{z_2} \frac{z_2^{x_2 - N(\mu_c - 1)}}{z_1^{x_1 - N(\mu_c - 1)}} e^{N(g_{\mu_c,\beta_0}(z_1) + g_{\mu_c,\beta_0}(1/z_2))} \frac{z_2}{z_1 - z_2}, \tag{3.60}
$$

where

$$z_1 \sim p_c(\beta_0)\left(1 - \frac{iw_1}{d(\beta_0)N^{1/3}}\right), \quad \frac{1}{z_2} \sim \frac{1}{p_c(\beta_0)}\left(1 - \frac{iw_2}{d(\beta_0)N^{1/3}}\right). \tag{3.61}$$

From (3.21)– (3.24) and noticing

$$\begin{aligned}&\frac{z_2^{x_2+N(1-\mu_c(\beta_0))}}{z_1^{x_1+N(1-\mu_c(\beta_0))}}\\&\sim p_c(\beta_0)^{d_G N^{\frac{1}{2}}(\xi_2-\xi_1)} \quad \exp\left[iw_1(\frac{d_G}{d}N^{\frac{1}{6}}\xi_1 + \frac{a_{G-}-a}{d}N^{\frac{2}{3}}) + iw_2(\frac{d_G}{d}N^{\frac{1}{6}}\xi_1 + \frac{a_{G-}-a}{d}N^{\frac{2}{3}})\right],\end{aligned} \tag{3.62}$$

we obtain

$$\begin{aligned}\tilde{K}_{2N} &\sim (p_c(\beta_0))^{d_G N^{\frac{1}{2}}(\xi_2-\xi_1)} \frac{1}{dN^{1/3}}\\&\int_0^\infty d\lambda \mathrm{Ai}(\frac{d_G}{d}N^{\frac{1}{6}}\xi_1 + \frac{a_{G-}-a}{d}N^{\frac{2}{3}} + \lambda)\mathrm{Ai}(\frac{d_G}{d}N^{\frac{1}{6}}\xi_2 + \frac{a_{G-}-a}{d}N^{\frac{2}{3}} + \lambda).\end{aligned} \tag{3.63}$$

Using the asymptotics of Airy function

$$\mathrm{Ai}(\frac{d_G}{d}N^{\frac{1}{6}}\xi_1 + \frac{a_{G-}-a}{d}N^{\frac{2}{3}} + \lambda) \sim \frac{1}{2\sqrt{\pi}}\frac{e^{-\frac{2}{3}N\frac{(a_{G-}-a)}{d}}}{\sqrt{\frac{a_{G-}-a}{d}N^{\frac{2}{3}}}}, \tag{3.64}$$

where $a_{G-} - a > 0$, we can find $\tilde{K}_{2N}$ goes to 0 asymptotically. Combining this fact and the relation

$$\begin{aligned}&\tilde{K}_{2N}(\beta_0 N, x_1; \beta_0 N, x_2)\\&\sim (1-\gamma_+\gamma_-)\mathcal{D}_{1+}(\beta_0,\xi_1)\mathcal{D}_{1-}(\beta_0,\xi_2)\\&\sim (1-\gamma_+\gamma_-)\left[\frac{1}{(\gamma_+\gamma_-)^{a_{G-}(\gamma_-)}}\frac{(1-\alpha/\gamma_-)^{1+\beta_0}}{(1-\alpha\gamma_-)^{1-\beta_0}}\frac{(1-\alpha/\gamma_+)^{1-\beta_0}}{(1-\alpha\gamma_+)^{1+\beta_0}} \times e^{-\frac{(a_G(\gamma_-)-a_G(\gamma_+))^2}{4(\beta_0-\beta_+)}\frac{d_G(\gamma_+)}{d_G(\gamma_-)}}\right]^N,\end{aligned} \tag{3.65}$$

one finds

$$\left[\frac{1}{(\gamma_+\gamma_-)^{a_{G-}(\gamma_-)}}\frac{(1-\alpha/\gamma_-)^{1+\beta_0}}{(1-\alpha\gamma_-)^{1-\beta_0}}\frac{(1-\alpha/\gamma_+)^{1-\beta_0}}{(1-\alpha\gamma_+)^{1+\beta_0}} \times e^{-\frac{(a_G(\gamma_-)-a_G(\gamma_+))^2}{4(\beta_0-\beta_+)}\frac{d_G(\gamma_+)}{d_G(\gamma_-)}}\right] < 1. \tag{3.66}$$

Next we consider the case where $\beta_1 \neq \beta_2$. From (3.59), it is straightforward to see

$$\begin{aligned}&\tilde{K}_{2N}(\beta_1 N, x_1; \beta_2 N, x_2)\\&\sim (1-\gamma_+\gamma_-)\mathcal{D}_{1+}(\beta_1,\xi_1)\mathcal{D}_{1-}(\beta_2,\xi_2)\\&\sim (1-\gamma_+\gamma_-) \times \gamma_-^{x_2-x_1}\left\{(1-\alpha/\gamma_-)(1-\alpha\gamma_-)\right\}^{\beta_1-\beta_2}\\&\qquad \times \left[\frac{1}{(\gamma_+\gamma_-)^{a_{G-}(\gamma_-)}}\frac{(1-\alpha/\gamma_-)^{1+\beta_2}}{(1-\alpha\gamma_-)^{1-\beta_2}}\frac{(1-\alpha/\gamma_+)^{1-\beta_2}}{(1-\alpha\gamma_+)^{1+\beta_2}} \times e^{-\frac{(a_G(\gamma_-)-a_G(\gamma_+))^2}{4(\beta_2-\beta_+)}\frac{d_G(\gamma_+)}{d_G(\gamma_-)}}\right]^N.\end{aligned} \tag{3.67}$$

Hence from (3.66) and (3.67) one finds that the contribution of $\tilde{K}_{2N}(\beta_1 N, x_1; \beta_2 N, x_2)$ is of order $\mathcal{O}(e^{-N})$ and is negligible in the Fredholm determinant of (3.49).

To obtain the asymptotics of (3.49), one also needs the asymptotics of ϕ. Let us represent $\phi_{r_1,r_2}(x_1,x_2)$ as

$$\phi_{r_1,r_2}(x_1,x_2) = \frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i}\int_{C_1}\frac{dz}{z}e^{Nf_{y_1,y_2}(z)}, \tag{3.68}$$

where

$$f_{y_1,y_2}(z) = g_{y_1+1,\beta_1}(z) - g_{y_2+1,\beta_2}(z). \tag{3.69}$$

We scale y_i as

$$y_i = a_{G-}(\beta_i,\gamma_-) + \frac{d_G(\gamma_-)\xi_i}{N^{\frac{1}{2}}} \equiv y_0 + \delta y_i, \tag{3.70}$$

and adjust the path of z such that it crosses the saddle point p_c^f of $f_{y_1,y_2}(z)$,

$$z = p_c^f\left(1 + \frac{iw}{(\beta_2-\beta_1)d_G N^{\frac{1}{2}}}\right) \sim \gamma_-\left(1 + \frac{1}{(\beta_2-\beta_1)d_G N^{\frac{1}{2}}}(\xi_1-\xi_2+iw)\right). \tag{3.71}$$

Then we get

$$f_{y_1,y_2}(z) \sim f_{y_0,y_0}(z_0) - \frac{d_G\log\gamma_-(\xi_1-\xi_2)}{N^{\frac{1}{2}}} - \frac{(\xi_1-\xi_2)^2+w^2}{2(\beta_2-\beta_1)}. \tag{3.72}$$

From (3.68) and (3.72), one finds

$$\phi_{r_1,r_2}(z) \sim \frac{\gamma_-^{x_2-x_1}}{d_G N^{\frac{1}{2}}}\left[\frac{(1-\alpha)^2}{(1-\alpha\gamma_-)(1-\alpha/\gamma_-)}\right]^{N(\beta_2-\beta_1)}\frac{e^{\frac{-(\xi_2-\xi_1)^2}{2(\beta_2-\beta_1)}}}{\sqrt{2\pi(\beta_2-\beta_1)}}. \tag{3.73}$$

Substituting these asymptotic forms to (3.49) and picking up the terms which do not vanish asymptotically, one finds

$$\begin{aligned}
&\mathbb{P}[h^+(r_1,M)<l_1,\cdots,h^+(r_m,M)<l_m] \\
&\sim \det[1+K_{2N}g]\left\{\sum_{k=1}^m(\gamma_+\gamma_-)^{l_k}\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}\big(-1+\Lambda(\beta,s)\big)\right\} \\
&\quad + 1 - \sum_{k=1}^m\int_{s_k}^\infty\frac{e^{-\frac{\xi_1^2}{2(\beta_--\beta_k)}}}{\sqrt{2\pi(\beta_--\beta_k)}}d\xi_1 \\
&\quad + \sum_{i=1}^m\sum_{k_1<\cdots<k_i}\int_{s_{k_1}}^\infty\cdots\int_{s_{k_i}}^\infty d\xi_1\cdots d\xi_i\frac{e^{\frac{-(\xi_2-\xi_1)^2}{2(\beta_2-\beta_1)}}}{\sqrt{2\pi(\beta_2-\beta_1)}}\cdots\frac{e^{\frac{-(\xi_i-\xi_{i-1})^2}{2(\beta_i-\beta_{i-1})}}}{\sqrt{2\pi(\beta_i-\beta_{i-1})}}\frac{e^{-\frac{\xi_i^2}{2(\beta_--\beta_i)}}}{\sqrt{2\pi(\beta_--\beta_i)}},
\end{aligned} \tag{3.74}$$

where $(\gamma_+\gamma_-)^{l_k}\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}\Lambda(\beta,s)$ represents the contribution of the terms included in $\sum D_-E'D_+$ such as

$$\begin{aligned}
&\sum_{k,k'}\sum_{x_1,x_2} D_{2-}(u_k,x_1)\phi_{2u_k,2u_{k'}}(x_1,x_2)D_{2+}(u_{k'},x_2),\\
&\sum_{k,k",k'}\sum_{x_1,x_2,x_3} D_{2-}(u_k,x_1)\phi_{2u_k,2u_{k"}}(x_1,x_2)\tilde{K}_{2N}(u_{k"},x_3;u_{k'},x_2)D_{2+}(u_{k'},x_2),\\
&\sum_{k,k",k'}\sum_{x_1,x_2,x_3} D_{2-}(u_k,x_1)\phi_{2u_k,2u_{k"}}(x_1,x_3)\phi_{2u_{k"},2u_{k'}}(x_3,x_2)D_{2+}(k',x_2),
\end{aligned}\tag{3.75}$$

and so on. In fact the first term in (3.74) cancels due to the subtraction in (3.36);

$$\begin{aligned}
&\det[1+K_{2N}g]\\
&\sim \det[1+K_{2N}g]\left\{\sum_{k=1}^m(\gamma_+\gamma_-)^{l_k}\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}\big(-1+\Lambda(\beta,s)\big)\right\}\\
&\qquad\qquad -\gamma_+\gamma_-\det[1+K_{2N}g_{l-1}]\left\{\sum_{k=1}^m(\gamma_+\gamma_-)^{l_k-1}\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}\big(-1+\Lambda(\beta,s)\big)\right\}\\
&\sim \frac{1}{d_G(\gamma_-)N^{\frac12}}\sum_{k=1}^m\left(\frac{\partial}{\partial s_k}\det[1+K_{2N}g]\right)\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}\big(-1+\Lambda(\beta,s)\big)(\gamma_+\gamma_-)^{l_k}\\
&\sim \frac{1}{d_G(\gamma_-)N^{\frac12}}\sum_{k=1}^m\tilde{K}_{2N}(u_k,x=a_{G-}N+d_GN^{\frac12}s_k;u_k,x=a_{G-}N+d_GN^{\frac12}s_k)\frac{F(u_k,1/\gamma_+)}{F(u_k,\gamma_-)}\\
&\quad\big(-1+\Lambda(\beta,s)\big)(\gamma_+\gamma_-)^{l_k}\\
&\sim 0,
\end{aligned}\tag{3.76}$$

where $g_{l-1}(r_j,x_j)=-\chi_{(l_j-1,\infty)(x_j)}$. Thus when we consider the asymptotic behavior of (3.36), $\mathbb{P}[h^+(r_1,M),\cdots,h^+(r_m,M)]$ can be replaced with

$$\begin{aligned}
&\mathbb{P}'[h^+(r_1,M),\cdots,h^+(r_m,M)]\\
&=1-\sum_{k=1}^m\int_{s_k}^\infty d\xi_k\frac{e^{-\frac{\xi_k^2}{2(\beta_--\beta_k)}}}{\sqrt{2\pi(\beta_--\beta_k)}}\\
&\quad+\sum_{i=1}^m\sum_{k_1<\cdots<k_i}\int_{s_{k_1}}^\infty\cdots\int_{s_{k_i}}^\infty d\xi_1\cdots d\xi_i\frac{e^{\frac{-(\xi_2-\xi_1)^2}{2(\beta_2-\beta_1)}}}{\sqrt{2\pi(\beta_2-\beta_1)}}\cdots\frac{e^{\frac{-(\xi_i-\xi_{i-1})^2}{2(\beta_i-\beta_{i-1})}}}{\sqrt{2\pi(\beta_i-\beta_{i-1})}}\frac{e^{-\frac{\xi_i^2}{2(\beta_--\beta_i)}}}{\sqrt{2\pi(\beta_--\beta_i)}}.
\end{aligned}\tag{3.77}$$

Using the fact that (3.36) can be expressed as

$$\begin{aligned}
&\mathbb{P}[h(r_1,M)<l_1,\cdots,h(r_m,M)<l_m] \\
&\sim \left(1+\frac{1}{(1-\gamma_+\gamma_-)d_G N^{\frac{1}{2}}}\sum_{k=1}^{m}\frac{\partial}{\partial s_k}\right)\mathbb{P}'[h^+(r_1,M)<l_1,\cdots,h^+(r_m,M)<l_m] \\
&\sim \mathbb{P}'[h^+(r_1,M)<l_1,\cdots,h^+(r_m,M)<l_m],
\end{aligned} \tag{3.78}$$

one finally gets

$$\begin{aligned}
&\mathbb{P}[h(r_1,M)<l_1,\cdots,h(r_m,M)<l_m] \\
&\sim 1-\sum_{k=1}^{m}\int_{s_k}^{\infty}d\xi_k\frac{e^{-\frac{\xi_k^2}{2(\beta_--\beta_k)}}}{\sqrt{2\pi(\beta_--\beta_k)}} \\
&\quad+\sum_{i=1}^{m}\sum_{k_1<\cdots<k_i}\int_{s_{k_1}}^{\infty}\cdots\int_{s_{k_i}}^{\infty}d\xi_1\cdots d\xi_i\frac{e^{\frac{-(\xi_2-\xi_1)^2}{2(\beta_2-\beta_1)}}}{\sqrt{2\pi(\beta_2-\beta_1)}}\cdots\frac{e^{\frac{-(\xi_i-\xi_{i-1})^2}{2(\beta_i-\beta_{i-1})}}}{\sqrt{2\pi(\beta_i-\beta_{i-1})}}\frac{e^{-\frac{\xi_i^2}{2(\beta_--\beta_i)}}}{\sqrt{2\pi(\beta_--\beta_i)}}.
\end{aligned} \tag{3.79}$$

This expression is the same as the Fredholm representation in (3.7) with the kernel (3.8). □

We end this chapter by providing a proof of (ii-a) in Theorem 2.1, for which the same strategy as the above proof of (i-c),(ii-a) and (iii-a) is applicable. The only difference is the asymptotics of D_{1-} (3.58). This changes to

$$D_{1-}(u_c=\beta_c N,x)\sim\mathcal{D}'_{1-}(\beta_c,\xi)\equiv\frac{1}{F(u_c,1/\gamma_+)(1-\gamma_+\gamma_-)\gamma_+^x}\frac{e^{-\frac{\xi^2}{2(\beta_c-\beta_+)}\frac{d_G(\gamma_-)^2}{d_G(\gamma_+)^2}}}{\sqrt{2\pi(\beta-\beta_+)}\frac{d_G(\gamma_+)}{d_G(\gamma_-)}d_G(\gamma_-)N^{\frac{1}{2}}}. \tag{3.80}$$

This means that we can obtain the Gaussian as a scaling limit for both $D_{1+}(u,x)$ and $D_{1-}(u,x)$ since the β_c point is the crossing point of two lines with Gaussian fluctuation. Together with this, the asymptotics of $\tilde{K}_{2N}$ also change;

$$\tilde{K}_{2N}(u_c,x_1;u_c,x_2)\sim(1-\gamma_+\gamma_-)\mathcal{D}_{1+}(\beta_c,\xi_1)\mathcal{D}'_{1-}(\beta_c,\xi_2)\sim\mathcal{O}(e^{-N}). \tag{3.81}$$

Considering these two relations in addition to (3.57) and (3.73), we pick up the terms

which do not vanish in (3.49),

$$
\begin{aligned}
&\mathbb{P}[h^+(r_c = 2u_c, M) < l] \\
&\sim \det[1 + gK_{2N}] \left\{ (\gamma_+\gamma_-)^l \frac{F(u_c, 1/\gamma_+)}{F(u_c, \gamma_-)} (-1 + \Lambda(\beta_c, s)) \right\} \\
&\quad + 1 - \int_s^\infty d\xi_1 \frac{e^{-\frac{\xi_1^2}{2(\beta_- - \beta_c)}}}{\sqrt{2\pi(\beta_- - \beta_c)}} - \int_s^\infty d\xi_2 \frac{e^{-\frac{\xi_2^2}{2(\beta_c - \beta_+)} \frac{d_G(\gamma_-)^2}{d_G(\gamma_+)^2}}}{\sqrt{2\pi(\beta_c - \beta_+)} \frac{d_G(\gamma_+)}{d_G(\gamma_-)}} \\
&\quad + \int_s^\infty d\xi_1 \frac{e^{-\frac{\xi_1^2}{2(\beta_- - \beta_c)}}}{\sqrt{2\pi(\beta_- - \beta_c)}} \int_s^\infty d\xi_2 \frac{e^{-\frac{\xi_2^2}{2(\beta_c - \beta_+)} \frac{d_G(\gamma_-)^2}{d_G(\gamma_+)^2}}}{\sqrt{2\pi(\beta_c - \beta_+)} \frac{d_G(\gamma_+)}{d_G(\gamma_-)}},
\end{aligned} \tag{3.82}
$$

Here we can neglect the first term since this term vanishes due to the subtraction in (3.36). Hence one obtains,

$$
\lim_{N\to\infty} \mathbb{P}[H_N^{(G_-)}(\beta_c, \gamma_-) \le s] = \int_{-\infty}^s d\xi_1 \frac{e^{-\frac{\xi_1^2}{2(\beta_- - \beta_c)}}}{\sqrt{2\pi(\beta_- - \beta_c)}} \int_{-\infty}^s d\xi_2 \frac{e^{-\frac{\xi_2^2}{2(\beta_c - \beta_+)} \frac{d_G(\gamma_-)^2}{d_G(\gamma_+)^2}}}{\sqrt{2\pi(\beta_c - \beta_+)} \frac{d_G(\gamma_+)}{d_G(\gamma_-)}}. \tag{3.83}
$$

This completes the proof of (ii-a) in Theorem 2.1. In principle, one can also apply the same method to the case of the multipoint function including the β_c point.

4 Transition around GOE^2

4.1 Limiting Kernel

Let us suppose that, when $\gamma_- = \gamma_0$, the limit shapes of $a(\beta)$ and $a_{G-}(\beta, \gamma_-)$ cross at $\beta = \beta_0$. We call this point the GOE^2 point.

In this section, we obtain the kernel describing the multi-point height fluctuation near this GOE^2 point. We consider the case where the parameter $\alpha \le \gamma_+ < 1/\gamma_0$ is fixed and γ_- scales like

$$
\gamma_- = \gamma_0 \left(1 - \frac{\omega}{d(\beta_0) N^{1/3}} \right), \tag{4.1}
$$

with ω fixed. The result is

Theorem 4.1.

$$
\lim_{N\to\infty} \mathbb{P}[H_N(\tau_1, \beta_0) \le s_1, \cdots, H_N(\tau_m, \beta_0) \le s_m] = \det(1 + \mathcal{K}\mathcal{G}), \tag{4.2}
$$

where $\mathcal{G}(\tau_j, \xi) = -\chi_{(s_j,\infty)}(\xi)$ *(*$j = 1, 2, \cdots, m$*) and*

$$
\begin{aligned}
&\mathcal{K}(\tau_1, \xi_1; \tau_2, \xi_2) \\
&= \begin{cases} \mathcal{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) + \mathrm{Ai}(\xi_1) \int_0^\infty d\lambda e^{-(\omega+\tau_2)\lambda} \mathrm{Ai}(\xi_2 - \lambda), & \omega + \tau_2 > 0, \\ \mathcal{K}_2(\tau_1, \xi_1; \tau_2, \xi_2) - \mathrm{Ai}(\xi_1) \int_0^\infty d\lambda e^{(\omega+\tau_2)\lambda} \mathrm{Ai}(\xi_2 + \lambda) + \mathrm{Ai}(\xi_1) e^{\frac{\tau_2^3 + \omega^3}{3} - \xi_2(\tau_2 + \omega)}, & \omega + \tau_2 < 0. \end{cases}
\end{aligned} \tag{4.3}
$$

Remark. This kernel seems to be new. If we set $\omega = 0$ from the beginning, this gives Theorem 3.1 (i-b). Notice that when we focus on the one-point correlation function, the case where $\tau = 0, \omega = \tau'$ is essentially the same as the case where $\tau = \tau', \omega = 0$.

Proof. First we give the contour integral representation of the kernel. For $\gamma_- < R_2$, it reads

$$\begin{aligned}
&\tilde{K}_N(r_1 = 2u_1, x_1; r_2 = 2u_2, x_2) \\
&= \frac{(1-\alpha)^{2(u_2-u_1)}}{(2\pi i)^2} \int_{C_{R_1}} \frac{dz_1}{z_1} \int_{C_{R_2}} \frac{dz_2}{z_2} \frac{z_2^{x_2}}{z_1^{x_1}} \frac{z_1}{z_1 - z_2} \frac{(1-\alpha/z_1)^{N-1+u_1}(1-\alpha z_2)^{N-1-u_2}}{(1-\alpha z_1)^{N-1-u_1}(1-\alpha/z_2)^{N-1+u_2}} \\
&\quad \times \frac{1-\gamma_-/z_1}{1-\gamma_-/z_2} \frac{1-\gamma_+ z_2}{1-\gamma_+ z_1}.
\end{aligned} \tag{4.4}$$

For $\gamma_- > R_2$, we have to add the contribution of the pole at $z_2 = \gamma_-$, resulting in

$$\begin{aligned}
&\tilde{K}_N(r_1 = 2u_1, x_1; r_2 = 2u_2, x_2) \\
&= \frac{(1-\alpha)^{2(u_2-u_1)}}{(2\pi i)^2} \int_{C_{R_1}} \frac{dz_1}{z_1} \int_{C_{R_2}} \frac{dz_2}{z_2} \frac{z_2^{x_2}}{z_1^{x_1}} \frac{z_1}{z_1 - z_2} \frac{(1-\alpha/z_1)^{N-1+u_1}(1-\alpha z_2)^{N-1-u_2}}{(1-\alpha z_1)^{N-1-u_1}(1-\alpha/z_2)^{N-1+u_2}} \\
&\quad \times \frac{1-\gamma_-/z_1}{1-\gamma_-/z_2} \frac{1-\gamma_+ z_2}{1-\gamma_+ z_1} \\
&\quad + \frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i} \int_{C_{R_1}} \frac{dz_1}{z_1} \frac{1}{z_1^{x_1}} \frac{(1-\alpha/z_1)^{N-1+u_1} \gamma_-^{x_2}}{(1-\alpha z_1)^{N-1-u_1}(1-\gamma_+ z_1)} \frac{(1-\alpha\gamma_-)^{N-1-u_2}(1-\gamma_+\gamma_-)}{(1-\alpha/\gamma_-)^{N-1+u_2}}.
\end{aligned} \tag{4.5}$$

Now we consider the asymptotics when we set (4.1),

$$x_i = Na(\beta_0) + d(\beta_0) N^{1/3} \xi_i, \tag{4.6}$$

$$r_i = 2u_i = 2N\left(\beta_0 + \frac{c(\beta_0)\tau_i}{N^{1/3}}\right), \tag{4.7}$$

with $i = 1, 2$ and take $N \to \infty$. The analysis is almost the same as in [25] and hence the details are omitted. When $\omega + \tau_2 > 0$, one uses (4.4) and deforms the contours of z_1 and z_2 such that

$$z_1 \sim \gamma_0 \left(1 + \frac{1}{d(\beta_0) N^{1/3}} (\tau_1 - i w_1)\right), \tag{4.8}$$

$$\frac{1}{z_2} \sim \frac{1}{\gamma_0} \left(1 - \frac{1}{d(\beta_0) N^{1/3}} (\tau_2 + i w_2)\right). \tag{4.9}$$

Then we get

$$K_N \sim (1-\alpha)^{2(u_2-u_1)}(p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}}\frac{1}{d(\beta_0)N^{1/3}}$$
$$\times e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1}$$
$$\times\frac{1}{4\pi^2}\int_{\mathrm{Im}w_1=\eta_1}dw_1\int_{\mathrm{Im}w_2=\eta_2}dw_2\left(-\frac{1}{\tau_2-\tau_1+i(w_1+w_2)}+\frac{1}{\omega+\tau_2+iw_2}\right)$$
$$\times e^{i\xi_1w_1+i\xi_2w_2+\frac{i}{3}(w_1^3+w_2^3)}, \tag{4.10}$$

with the definition of λ_i given in (3.23). Noticing

$$\frac{1}{4\pi^2}\int_{\mathrm{Im}w_1=\eta_1}dw_1\int_{\mathrm{Im}w_2=\eta_2}dw_2\left(-\frac{1}{\tau_2-\tau_1+i(w_1+w_2)}+\frac{1}{\omega+\tau_2+iw_2}\right)$$
$$\times e^{i\xi_1w_1+i\xi_2w_2+\frac{i}{3}(w_1^3+w_2^3)}$$
$$=\int_0^\infty e^{-\lambda(\tau_1-\tau_2)}\mathrm{Ai}(\xi_1+\lambda)\mathrm{Ai}(\xi_2+\lambda)d\lambda+\mathrm{Ai}(\xi_1)\int_0^\infty d\lambda e^{-(\omega+\tau_2)\lambda}\mathrm{Ai}(\xi_2-\lambda), \tag{4.11}$$

one gets the desired expression for this case.

The $\omega+\tau_2<0$ case can also be treated in a similar fashion. In this case we consider the asymptotics using (4.5) since the contour of z_2 (4.9) cross the real axis on the left of γ_0. The second term in the RHS of (4.5) produces the third term in the RHS of (4.3). □

4.2 Connection to the Baik-Rains analysis

When we specialize to the case of the one-point height fluctuation, our formula reduces to

$$\lim_{N\to\infty}\mathbb{P}[H_N(0,\beta_0)\le s]=\det(1+\mathcal{K}\mathcal{G}), \tag{4.12}$$

where $\mathcal{G}(\xi)=-\chi_{(s,\infty)}(\xi)$ and

$$\mathcal{K}(x,y)=\mathcal{K}_2(x,y)+A(x)B(y,\omega), \tag{4.13}$$

with

$$\mathcal{K}_2(x,y)=\int_0^\infty\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)d\lambda, \tag{4.14}$$

$$A(x)=\mathrm{Ai}(x), \tag{4.15}$$

$$B(x,\omega)=\begin{cases}\displaystyle\int_0^\infty d\lambda e^{-\omega\lambda}\mathrm{Ai}(x-\lambda), & \omega>0 \quad (4.16a)\\ \displaystyle-\int_0^\infty d\lambda e^{\omega\lambda}\mathrm{Ai}(x+\lambda)+e^{\omega^3/3-x\omega}, & \omega<0. \quad (4.16b)\end{cases}$$

Notice that the expression on the right hand side of (4.12) is independent of β_0 and that, for the special case where $\beta_0=0$, another expression for the same quantity was previously

obtained in [27]. Hence the two expressions should be the same. For $\omega = 0$, this was already shown in [32]. In this subsection, we prove the equivalence for any ω generalizing the arguments in [32].

We first proceed as

$$\begin{aligned}\det(1+\mathcal{K}\mathcal{G}) &= \det(1+(\mathcal{K}_2 + A\otimes B)\mathcal{G}) \\ &= \det(1+\mathcal{K}_2\mathcal{G})\det(1+(1+\mathcal{K}_2\mathcal{G})^{-1}A\otimes B\mathcal{G}) \\ &= F_2(s)\left(1-\int_s^\infty dx\int_s^\infty dy\rho(x,y)A(x)B(y,\omega)\right),\end{aligned} \tag{4.17}$$

where $\rho(x,y)$ is the kernel of the operator $(1+\mathcal{K}_2\mathcal{G})^{-1}$. Hence if one defines

$$a(s,\omega) = 1-\int_s^\infty dx\int_s^\infty dy\rho(x,y)A(x)B(y,\omega), \tag{4.18}$$

one has

$$\det(1+\mathcal{K}\mathcal{G}) = F_2(s)a(s,\omega) \equiv F(s). \tag{4.19}$$

Let us also define

$$b(s,\omega) = \int_s^\infty dy\rho(s,y)B(y,\omega), \tag{4.20}$$

and

$$Q(x) = \int_s^\infty dy\rho(x,y)A(y), \tag{4.21}$$

$$q(s) = Q(s). \tag{4.22}$$

Now we show that the functions a, b have the following properties.

Proposition 4.2.

$$\frac{\partial}{\partial s}a = qb, \tag{4.23}$$

$$\frac{\partial}{\partial s}b = qa - \omega b, \tag{4.24}$$

$$\frac{\partial}{\partial \omega}a = q^2 a - (q' + \omega q)b, \tag{4.25}$$

$$\frac{\partial}{\partial \omega}b = (q' - \omega q)a + (\omega^2 - s - q^2)b. \tag{4.26}$$

Proof. From the results in [5], one has

$$R(x,y) \equiv \rho(x,y) - \delta^+(x-y) = \frac{Q(x)P(y) - P(x)Q(y)}{x-y}, \tag{4.27}$$

$$\frac{\partial}{\partial s} Q(y) = q(s)\big(\delta^+(y-s) - \rho(s,y)\big), \tag{4.28}$$

$$\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \rho(x,y) = -Q(x)Q(y), \tag{4.29}$$

$$Q'(y) = P(y) - u(s)Q(y) + q(s)\rho(s,y) - q(s)\delta^+(y-s), \tag{4.30}$$

$$P'(y) = yQ(y) - 2Q(y)v(s) + u(s)P(y) + p(s)\rho(s,y) - p(s)\delta^+(y-s), \tag{4.31}$$

$$u^2 - 2v = q^2, \tag{4.32}$$

$$q' = p - qu, \tag{4.33}$$

where

$$P(x) = \int_s^\infty \rho(x,y)\mathrm{Ai}'(y)dy, \tag{4.34}$$

$$p(s) = P(s), \tag{4.35}$$

$$u(s) = \int_s^\infty Q(x)\mathrm{Ai}(x)dx, \tag{4.36}$$

$$v(s) = \int_s^\infty P(x)\mathrm{Ai}(x)dx, \tag{4.37}$$

$$\int_s^\infty \delta^+(y-s)f(y)dy = f(s). \tag{4.38}$$

Besides, we also use

$$\frac{\partial}{\partial y} B(y,\omega) = \mathrm{Ai}(y) - \omega B(y,\omega), \tag{4.39}$$

$$\frac{\partial}{\partial \omega} B(y,\omega) = \mathrm{Ai}'(y) - \omega \mathrm{Ai}(y) + (\omega^2 - y)B(y,\omega). \tag{4.40}$$

These equations can be shown immediately. For the case where $\omega > 0$ we use (4.16a) and compute

$$\frac{\partial}{\partial y} B(y,\omega) = \int_0^\infty d\lambda e^{-\omega\lambda} \frac{\partial \mathrm{Ai}(y-\lambda)}{\partial y} = -\int_0^\infty d\lambda e^{-\omega\lambda} \frac{\partial \mathrm{Ai}(y-\lambda)}{\partial \lambda}, \tag{4.41}$$

$$\begin{aligned}\frac{\partial}{\partial \omega} B(y,\omega) &= -\int_0^\infty d\lambda e^{\omega\lambda} \lambda \mathrm{Ai}(y-\lambda) \\ &= -\int_0^\infty d\lambda e^{-\omega\lambda} \big\{ y\mathrm{Ai}(y-\lambda) - \mathrm{Ai}''(y-\lambda) \big\}. \end{aligned} \tag{4.42}$$

In (4.42) the Airy equation, $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, is used. These equations lead to (4.39) and (4.40) respectively. For the case where $\omega < 0$ they can be shown by applying the same method to (4.16b).

Now the first two equalities, (4.23) and (4.24), can be shown as

$$\begin{aligned}\frac{\partial}{\partial s}a &= Q(s)B(s,\omega) - \int_s^\infty dy \frac{\partial}{\partial s}Q(y)B(y,\omega) \\ &= q(s)\int_s^\infty dy\rho(s,y)B(y,\omega) = qb, \qquad (4.43)\\ \frac{\partial}{\partial s}b &= -\int_s^\infty dy\left(\frac{\partial}{\partial y}\rho(s,y)\right)B(y,\omega) - q(s)\int_s^\infty dyQ(y)B(y,\omega) - \rho(s,s)B(s,\omega) \\ &= qa - q + \int_s^\infty dy\rho(s,y)\frac{\partial}{\partial y}B(y,\omega) \\ &= qa - \omega b. \qquad (4.44)\end{aligned}$$

For the third equality (4.25), one starts from

$$\begin{aligned}\frac{\partial}{\partial \omega}a &= -\int_s^\infty Q(y)\frac{\partial}{\partial \omega}B(y,\omega)dy \\ &= -\int_s^\infty Q(y)\mathrm{Ai}'(y)dy + \omega\int_s^\infty Q(y)\mathrm{Ai}(y)dy \\ &\quad - \omega^2\int_s^\infty Q(y)B(y,\omega)dy + \int_s^\infty yQ(y)B(y,\omega)dy. \qquad (4.45)\end{aligned}$$

The second and the third terms are easily seen to be ωu and $-\omega^2(1-a)$ respectively. Using (4.30), the first term is computed as

$$\begin{aligned}-\int_s^\infty Q(y)\mathrm{Ai}'(y)dy &= -\big[Q(y)\mathrm{Ai}(y)\big]_s^\infty + \int_s^\infty Q'(y)\mathrm{Ai}(y)dy \\ &= \int_s^\infty P(y)\mathrm{Ai}(y)dy - u\int_s^\infty Q(y)\mathrm{Ai}(y)dy + q\int_s^\infty \rho(s,y)\mathrm{Ai}(y)dy \\ &= v - u^2 + q^2. \qquad (4.46)\end{aligned}$$

Next we consider the last term. First we calculate

$$\begin{aligned}yQ(y) &= P'(y) - uQ'(y) - Q(y)(u^2-2v) + (\rho-\delta^+)(qu-p) \\ &= P'(y) - uQ'(y) - Q(y)q^2 - \rho q' + \delta^+ q', \qquad (4.47)\end{aligned}$$

where we use (4.30), (4.31) in the first equality and (4.32), (4.33) in the second equality. Then one can show that the last term is

$$\begin{aligned}\int_s^\infty dyyQ(y)B(y,\omega) &= -q^2\int_s^\infty dyQ(y)B(y,w) - q'b + \int_s^\infty dyP'(y)B(y) \qquad (4.48)\\ &\quad - u\int_s^\infty dyQ'(y)B(y,w) + q'B(s,w) \\ &= q^2 a - (q'+\omega q)b - v + u^2 - q^2 - \omega u + \omega^2(1-a), \qquad (4.49)\end{aligned}$$

where we use (4.39) and (4.40). Combining these, one gets (4.25).

Finally, for the proof of the last equality (4.26), one starts from

$$\begin{aligned}\frac{\partial}{\partial \omega} b &= \int_s^\infty \rho(s,y) \frac{\partial}{\partial \omega} B(y,\omega) dy \\ &= \int_s^\infty dy \rho(s,y) \mathrm{Ai}'(y) - \omega \int_s^\infty dy \rho(s,y) \mathrm{Ai}(y) \\ &\quad + \omega^2 \int_s^\infty dy \rho(s,y) B(s,\omega) - \int_s^\infty dy y \rho(s,y) B(s,y). \end{aligned} \tag{4.50}$$

The first three terms are easily seen to be $p = q' + uq, -\omega q, \omega^2 b$ respectively. The last term can also be calculated. The term $y\rho$ in the integrand can be shown as

$$y\rho(s,y) = (p - qu)Q(y) - qQ'(y) + (q^2 + s)\rho(s,y) - q^2 \delta^+(s-y) \tag{4.51}$$

due to (4.27) and (4.30). Thus from (4.33), (4.39) and (4.51), one gets

$$-\int_s^\infty dy y \rho(s,y) B(s,y) = -(q' - \omega q) \int_s^\infty dy Q(y) B(y) - (q^2 + s) b - uq \tag{4.52}$$

These equations lead to (4.26). □

This proposition, combined with $a(s,0) = b(s,0) = e^{-\int_s^\infty q(x)dx}$ proved in [32], shows that the functions a, b as defined by (4.18),(4.20) are the same as those in [27] with the identification $a_{\mathrm{BR}} = a, b_{\mathrm{BR}} = -b, u = -q, w = 2\omega$. Thus the one-point height fluctuation defined in (4.19) corresponds to that in [27] under the above identification.

4.3 GOE^2 to GUE/Gaussian Transition

Using the above results, it is possible to study the fluctuation properties of the PNG model quite in detail. In this subsection, we set $\omega = 0$ and consider the transition from GOE^2 ($\tau = 0$) to GUE ($\tau \to \infty$) and Gaussian ($\tau \to -\infty$). As an example, we here consider the average of the scaled height. For the case where $\tau = 0$ or $\tau = \infty$, we can easily compute the average numerically using the Painlevé expression of the GOE^2 or GUE [5,6]. For the other values of τ, however, such a representation has not been known. To compute the value numerically, one can use the differential equations of a and b, (4.23)–(4.26). On the other hand, we also obtain the asymptotic behaviors of the average for each case where $\tau \sim 0, \infty, -\infty$ as follows.

- $\tau \sim 0$

The equations (4.23)–(4.26) enable us to know the behaviors of a for $\tau \sim 0$. We get

$$a(s,\tau) = a(s,0) + \left.\frac{\partial a(s,\tau)}{\partial \tau}\right|_{\tau=0} \tau + \frac{1}{2} \left.\frac{\partial^2 a(s,0)}{\partial \tau^2}\right|_{\tau=0} \tau^2 + \frac{1}{3!} \left.\frac{\partial^3 a(s,\tau)}{\partial \tau^3}\right|_{\tau=0} \tau^3 + \cdots . \tag{4.53}$$

hen the distribution behaves as Substituting (4.53) to $F(s)$ defined in (4.19), the average behaves as

$$\int sF'(s)ds \sim \int sF'_{\mathrm{GOE}^2}(s)ds + \tau \int e_1(s)ds + \frac{\tau^2}{2!}\int e_2(s)ds + \frac{\tau^3}{3!}\int e_3(s)ds + \cdots$$
$$= -0.49364 - 0.89941\tau + 0.41582\tau^2 - 0.12409\tau^3 + \cdots . \tag{4.54}$$

Here $e_i(s)$'s are computed as

$$e_1(s) = se^{-\int_s^\infty q(x)dx}\left\{P_2(s)(q(s)^2 - q(s)') - F_2(s)\left(q(s)^3 + sq(s) - q'(s)q(s)\right)\right\}, \tag{4.55}$$

$$e_2(s) = se^{-\int_s^\infty q(x)dx}\Big\{P_2(s)\Big(q(s)\ \Big(-q(s) + q(s)^4 + 2\,q(s)^2\,s + s^2 - s\,q'(s) - q'(s)^2\Big)\Big)$$
$$+F_2(s)\left(-q(s) + q(s)^4 + s\,q'(s) - q'(s)^2\right)\Big\}, \tag{4.56}$$

$$e_3(s) = se^{-\int_s^\infty q(x)dx}\Big\{P_2(s)\left(q(s)^3 + q(s)^6 + 2\,q(s)\,s - \left(q(s) + q(s)^4 + q(s)^2\,s + s^2\right)\,q'(s)\right.$$
$$\left. + \left(-q(s)^2 + s\right)\,q'(s)^2 + q'(s)^3\right)$$
$$+ F_2(s)\left(q(s)\ \left(2 - q(s)^3 - q(s)^6 + q(s)\,s - 3\,q(s)^4\,s - 3\,q(s)^2\,s^2 - s^3 + \right.\right.$$
$$\left.\left.\left(q(s) + q(s)^4 + q(s)^2\,s + s^2\right)\,q'(s) + \left(q(s)^2 + 2\,s\right)\,q'(s)^2 - q'(s)^3\right)\right)\Big\}, \tag{4.57}$$

using (4.23)– (4.26) and $a(s,0) = b(s,0) = e^{-\int_s^\infty q(x)dx}$ proved in [32].

- $\tau \to \infty$

One can also get the asymptotic behavior of a as $\tau \to \infty$. By noting

$$B(x,\tau) = \frac{1}{\tau}\int_0^\infty d\theta e^{-\theta}\mathrm{Ai}(x - \theta/\tau)$$
$$= \sum_{j=0}^\infty \frac{(-1)\mathrm{Ai}^{(j)}(x)}{\tau^{j+1}}\int_0^\infty d\theta\theta^j e^{-\theta}$$
$$\sim \frac{\mathrm{Ai}(x)}{\tau} - \frac{\mathrm{Ai}'(x)}{\tau^2} + \frac{\mathrm{Ai}''(x)}{\tau^3} - \cdots , \tag{4.58}$$

one finds

$$a(x,\tau) \sim 1 - \frac{1}{\tau}\int_s^\infty dx \int_s^\infty dy\rho(x,y)\mathrm{Ai}(x)\mathrm{Ai}(y) = 1 - \frac{u(s)}{\tau}, \tag{4.59}$$

where $u(s)$ defined in (4.36) reads

$$u(s) = \int_s^\infty q(x)^2 dx. \tag{4.60}$$

Then one has

$$\int sF'(s)ds \sim \int sF_2'(s)ds - \frac{1}{\tau}\int F_2(s)u(s)ds \sim -1.77109 + \frac{1}{\tau}. \tag{4.61}$$

- $\tau \to -\infty$

Finally we consider the asymptotics where $\tau \sim -\infty$. Due to (4.23), the probability density is represented as

$$\frac{dF(s)}{ds} = q(s)b(s,\tau)F_2(s) + a(s,\tau)F_2'(s). \tag{4.62}$$

We evaluate the first term of the above equation as follows. From the results of Baik-Rains,

$$\begin{aligned} &\lim_{\tau\to\infty} a(s,\tau) = 1, \\ &\lim_{\tau\to-\infty} b(s,\tau) = 0, \\ &a(s,\tau) = b(s,-\tau)e^{\tau^3 - s\tau}, \end{aligned} \tag{4.63}$$

one finds when $\tau \to \infty$,

$$b(s,\tau) \sim e^{\tau^3 - s\tau}. \tag{4.64}$$

Let us scale the variable s as

$$s = \tau^2 y, \tag{4.65}$$

and we assume $y > 0$. (This can be justified in the below discussion.) Using the asymptotic behavior of the Airy function, we also find

$$q(s) \sim \mathrm{Ai}(s) \sim \frac{1}{2\sqrt{\pi}s^{1/4}} e^{-\frac{2}{3}s^{\frac{3}{2}}}. \tag{4.66}$$

Then due to (4.64), (4.66) and $F_2(s) \sim 1$ as $s \to \infty$, one finds

$$q(s)b(s,\tau)F_2(s) \sim \frac{1}{2\sqrt{\pi}|\tau|^{1/2}y^{1/4}} e^{\tau^3 g(y)}, \tag{4.67}$$

where

$$g(y) = \frac{2}{3}y^{3/2} - y + 1. \tag{4.68}$$

The condition $g'(y) = 0$ leads to $y = 1$. Rescaling y as $y = 1 + y'/|\tau|^{3/2}$ and expanding $g(y)$ around $y = 1$, one gets

$$q(s)b(s,\tau)F_2(s) \sim \frac{1}{2\sqrt{\pi}|\tau|^{1/2}} e^{\frac{y'^2}{4}}. \tag{4.69}$$

On the other hand, for the second term, one finds

$$a(s,\tau)F_2'(s) = a(\tau^2 y,\tau)F_2'(\tau^2 y) \sim 0 \tag{4.70}$$

since $a(s,\tau) \sim 1, F_2'(s) \to 0$ as $s \to \infty$. Thus the probability density is approximated by the Gaussian with the peak at $s = \tau^2$ when $-\tau$ is large sufficiently and one gets

$$\int sF_2'(s)ds \sim \tau^2. \tag{4.71}$$

Using (4.54) , (4.61) and (4.71), we can calculate the average of the distribution numerically. This is shown in Fig 3. In principle, other statistical quantities such as two point function analyzed in [33] can be also studied.

Before closing the section, we would like to mention that the GOE^2 distribution seems rather universal [34] but the numerical values of statistical quantities of the GOE^2 have not been given in the literature. Since we have the data at hand, we list them here. The known data for GOE/GUE/GSE [36] are also presented for comparison. (In this table, the data for GSE is shown according to the notation in [18]. Therefore the values of the average and standard deviation in GSE are different from those values written in [35].)

	average	s.d.	skewness	kurtosis
GOE^2	-0.49364	1.1100	0.3917	0.309
GOE	-1.20653	1.2680	0.2935	0.165
GUE	-1.77109	0.9018	0.2241	0.093
GSE	-3.26243	1.0176	0.1655	0.049

5 Transition around F_0

5.1 Limiting Kernel

Let us suppose that, when $\gamma_- = \gamma_0 = 1/\gamma_+$, the limit shape is tangent to that in bulk (2.6) at $\beta = \beta_0$. Let us call this point the F_0 point. In this case we again have to consider the modified model as in the proof of (i-c), (ii-a) and (iii-a) in Theorem 3.1 since the parameter of geometric distribution at the $(i = 1, j = 1)$ point is close to unity in the original model. The relation between the modified and original models is given in (3.36),

$$\begin{aligned}
&\mathbb{P}[h(r_1, t = 2N) < l_1, \cdots, h(r_m, t = 2N) < l_m] \\
&= \mathbb{P}[h^+(r_1, 2N) < l_1, \cdots, h^+(r_m, 2N) < l_m] + \frac{\gamma_+\gamma_-}{1-\gamma_+\gamma_-}\big\{\mathbb{P}[h^+(r_1, 2N) < l_1, \cdots, h^+(r_m, 2N) < l_m] \\
&\quad - \mathbb{P}[h^+(r_1, 2N) < l_1 - 1, \cdots, h^+(r_m, 2N) < l_m - 1]\big\}. \qquad (5.1)
\end{aligned}$$

When we consider the case where $\gamma_+\gamma_- = 1$ we always treat $\mathbb{P}[h^+(r_1, t) < l_1, \cdots, h^+(r_m, t) < l_m]$ in the right hand side as the Fredholm determinant (3.10) with the kernel (3.11)– (3.13). In the scaling limit where

$$\begin{aligned}
r_i &= 2\beta_0 N + 2\tau_i N^{2/3}, \\
l_i &= a\left(\beta_0 + \frac{c(\beta_0)\tau_i}{N^{1/3}}\right) N + d(\beta_0) N^{\frac{1}{3}} s_i, \\
\gamma_\pm &= \frac{1}{\gamma_0}\left(1 - \frac{\omega_\pm}{d(\beta_0)N^{1/3}}\right), \qquad (5.2)
\end{aligned}$$

one finds

$$\lim_{N\to\infty}\mathbb{P}[H_N(\tau_1,\beta_0)\le s_1,\cdots,H_N(\tau_m,\beta_0)\le s_m]$$
$$=\left(1+\frac{1}{\omega_++\omega_-}\sum_{j=1}^m\frac{\partial}{\partial s_j}\right)\lim_{N\to\infty}\mathbb{P}[H_N^+(\tau_1,\beta_0)\le s_1,\cdots,H_N^+(\tau_m,\beta_0)\le s_m]. \tag{5.3}$$

We obtain the kernel describing the multi-point height fluctuation near this F_0 point. The result is

Theorem 5.1.
When $\omega_++\omega_->0$,

$$\lim_{N\to\infty}\mathbb{P}[H_N(\tau_1,\beta_0)\le s_1,\cdots,H_N(\tau_m,\beta_0)\le s_m]=\left(1+\frac{1}{\omega_++\omega_-}\sum_{j=1}^m\frac{\partial}{\partial s_j}\right)\det(1+\mathcal{K}\mathcal{G}), \tag{5.4}$$

where $\mathcal{G}(\tau_j,\xi)=-\chi_{(s_j,\infty)}(\xi)$ $(j=1,2,\cdots,m)$,

$$\mathcal{K}(\tau_1,\xi_1;\tau_2,\xi_2)=\mathcal{K}_2(\tau_1,\xi_1;\tau_2,\xi_2)+(\omega_++\omega_-)\mathcal{B}(\omega_+,\tau_1,\xi_1)\mathcal{B}'(w_-,\tau_2,\xi_2), \tag{5.5}$$

and

$$\mathcal{B}(\omega_+,\tau_1,\xi_1)=\begin{cases}\int_0^\infty d\lambda e^{-(\omega_+-\tau_1)\lambda}\mathrm{Ai}(\xi_1-\lambda), & \omega_+-\tau_1>0,\\ -\int_0^\infty d\lambda e^{(\omega_+-\tau_1)\lambda}\mathrm{Ai}(\xi_1+\lambda)+e^{\frac{-\tau_1^3+\omega_+^3}{3}-\xi_1(\omega_+-\tau_1)}, & \omega_+-\tau_1<0,\end{cases} \tag{5.6}$$

$$\mathcal{B}'(\omega_-,\tau_2,\xi_2)=\begin{cases}\int_0^\infty d\lambda e^{-(\omega_-+\tau_2)\lambda}\mathrm{Ai}(\xi_2-\lambda), & \omega_-+\tau_2>0,\\ -\int_0^\infty d\lambda e^{(\omega_-+\tau_2)\lambda}\mathrm{Ai}(\xi_2+\lambda)+e^{\frac{\tau_2^3+\omega_-^3}{3}-\xi_2(\omega_-+\tau_2)}, & \omega_-+\tau_2<0.\end{cases} \tag{5.7}$$

Remarks.

1. The Fredholm representation (5.4) and (5.5) includes the terms which are not analytic for $\omega_++\omega_-<0$, i.e., the difficulty of divergence that we discussed in the proof of Theorem 3.1 (i-c), (ii-a) and (iii-a) arises. However, one can avoid this in a similar manner as for the Theorem 3.1.

   For example let us consider the term included in this Fredholm determinant which corresponds to the last term in (3.44),

$$(\omega_++\omega_-)e^{\omega_+^3+\omega_-^3}\int_s^\infty d\xi e^{-\xi(\omega_++\omega_-)}. \tag{5.8}$$

   For $\omega_++\omega_->0$, this can be calculated as

$$(\omega_++\omega_-)e^{\omega_+^3+\omega_-^3}\int_s^\infty d\xi e^{-\xi(\omega_++\omega_-)}=e^{\omega_+^3+\omega_-^3-(\omega_++\omega_-)s}. \tag{5.9}$$

Here though the LHS is meaningful only for $\omega_+ + \omega_- > 0$, the RHS is analytic for all region of $\omega_+ + \omega_-$. Hence the difficulty of divergence of (5.8) can be avoided if one replaces the integral by the RHS in (5.9) when $\omega_+ + \omega_- < 0$. Other terms which diverge when $\omega_+ + \omega_- < 0$ can be treated in the same manner.

2. When we set $\omega_\pm = 0$ we obtain Theorem 3-1 (iii).

Proof. For the case where $\omega_+ - \tau_1 > 0, \omega_- + \tau_2 > 0$, the contour integral representation of the kernel reads

$$K_N(2u_1, x_1; 2u_2, x_2) = \frac{(1-a)^{2(u_2-u_1)}}{(2\pi i)^2} \int_{C_{R_1}} \frac{dz_1}{z_1} \int_{C_{R_2}} \frac{dz_2}{z_2} \frac{z_2^{x_2}}{z_1^{x_1}} \frac{z_1}{z_1 - z_2} \frac{(1-\alpha/z_1)^{N-1+u_1}(1-\alpha z_2)^{N-1-u_2}}{(1-\alpha z_1)^{N-1-u_1}(1-\alpha/z_2)^{N-1+u_2}} \times \frac{1-\gamma_+ z_2}{1-\gamma_+ z_1} \frac{1-\gamma_-/z_1}{1-\gamma_-/z_2}. \tag{5.10}$$

The asymptotics can be studied in the same manner as in the previous section with the result,

$$\begin{aligned} K_N(2u_1, x_1; 2u_2, x_2) \sim\ & (1-\alpha)^{2(u_2-u_1)} \big(p_c(\beta_0)\big)^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} \frac{1}{d(\beta_0)N^{1/3}} \\ & \times e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1} \\ & \times \frac{1}{4\pi^2} \int_{\mathrm{Im}w_1=\eta_1} dw_1 \int_{\mathrm{Im}w_2=\eta_2} dw_2 e^{i\xi_1 w_1 + i\xi_2 w_2 + \frac{i}{3}(w_1^3+w_2^3)} \\ & \times \left(-\frac{1}{\tau_2-\tau_1+i(w_1+w_2)} + \frac{\omega_+ + \omega_-}{(\omega_+ - \tau_1 + iw_1)(\omega_- + \tau_2 + iw_2)} \right). \end{aligned} \tag{5.11}$$

The one arrives at the desired expression.

In the case of $\omega_+ - \tau_1 > 0, \omega_- + \tau_2 < 0$ the contribution of the pole at $z_2 = \gamma_-$ are added as

$$\begin{aligned} & \frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i} \int_{C_{R_1}} \frac{dz_1}{z_1} \frac{1}{z_1^{x_1}} \frac{(1-\alpha/z_1)^{N-1+u_1}\gamma_-^{x_2}}{(1-\alpha z_1)^{N-1-u_1}(1-\gamma_+ z_1)} \frac{(1-\alpha\gamma_-)^{N-1-u_2}(1-\gamma_+\gamma_-)}{(1-\alpha/\gamma_-)^{N-1+u_2}} \\ & \sim (1-\alpha)^{2(u_2-u_1)} (p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} \frac{1}{d(\beta_0)N^{1/3}} \\ & \quad \times e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1} \\ & \quad \times e^{\frac{\tau_2^3+\omega_-^3}{3} - \xi_2(\tau_2+\omega_-)} \frac{1}{2\pi} \int_{\mathrm{Im}w_1=\eta_1} dw_1 \frac{e^{i\xi_1 w_1 + \frac{i}{3}w_1^3}}{\omega_+ - \tau_1 + iw_1} \end{aligned} \tag{5.12}$$

(5.11) and (5.12) lead to the desired expression. Similar to this case we add the term of

pole at $z_1 = 1/\gamma_+$ in the case of $\omega_+ - \tau_1 < 0, \omega_- + \tau_2 > 0$. This term reads

$$\begin{aligned}
&\frac{(1-\alpha)^{2(u_2-u_1)}}{2\pi i}\int_{C_{R_2}} \frac{dz_2}{z_2} z_2^{x_2} \frac{(1-\alpha z_2)^{N-1-u_2}\gamma_+^{x_1}}{(1-\alpha/z_2)^{N-1+u_2}(\gamma_+ z_2 - 1)} \frac{(1-\alpha\gamma_+)^{N-1+u_1}(1-\gamma_+\gamma_-)}{(1-\alpha/\gamma_+)^{N-1-u_1}} \\
&\sim (1-\alpha)^{2(u_2-u_1)}(p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} \frac{1}{d(\beta_0)N^{1/3}} \\
&\quad \times e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1} \\
&\quad \times e^{\frac{-\tau_1^3+\omega_+^3}{3}-\xi_1(\omega_+-\tau_1)} \frac{1}{2\pi}\int_{\mathrm{Im} w_2 = \eta_2} dw_2 \frac{e^{i\xi_1 w_2 + \frac{i}{3}w_2^3}}{\omega_- + \tau_2 + i w_2}
\end{aligned} \tag{5.13}$$

Finally we consider the case of $\omega_+ - \tau_1 < 0, \omega_- + \tau_2 < 0$. In addition to (5.11), (5.12) and (5.13) we have to consider the term caused by the product of the residues,

$$\begin{aligned}
&(1-\alpha)^{2(u_2-u_1)}\gamma_+^{x_1}\gamma_-^{x_2}\frac{(1-\alpha\gamma_+)^{N-1+u_1}}{(1-\alpha/\gamma_+)^{N-1-u_1}}\frac{(1-\alpha\gamma_-)^{N-1-u_2}}{(1-\alpha/\gamma_-)^{N-1+u_2}} \\
&\sim (1-\alpha)^{2(u_2-u_1)}(p_c(\beta_0))^{(\xi_2-\xi_1)d(\beta_0)N^{1/3}} \frac{1}{d(\beta_0)N^{1/3}} \\
&\quad \times e^{\lambda_1(\beta_0)c(\beta_0)N^{2/3}(\tau_1-\tau_2)+\lambda_2(\beta_0)c(\beta_0)^2N^{1/3}(\tau_1^2-\tau_2^2)+\lambda_3(\beta_0)c(\beta_0)^3(\tau_1^3-\tau_2^3)+\xi_2\tau_2-\xi_1\tau_1} \\
&\quad \times e^{\frac{\tau_2^3+\omega_-^3}{3}-\xi_2(\tau_2+\omega_-)} e^{\frac{-\tau_1^3+\omega_+^3}{3}-\xi_2(\omega_+-\tau_1)}.
\end{aligned} \tag{5.14}$$

Combining each term we get the desired expression. □

5.2 Connection to the Baik-Rains analysis

When we specialize to the case of the one-point height fluctuation, our formula again should be equivalent to the ones obtained in [27]. We show the equivalence in this subsection.

When we set $\tau_i = 0$, the Fredholm determinant reads

$$\det(1+\mathcal{K}\mathcal{G}) = F_2(s)\left\{1 - (\omega_+ + \omega_-)\int_s^\infty dx \int_s^\infty dy \rho(x,y)B(x,\omega_+)B(y,\omega_-)\right\}, \tag{5.15}$$

where $F_2(s)$ is the GUE Tracy-Widom distribution and $B(x,\omega)$ is defined in (4.16). We have the following proposition for this expression.

Proposition 5.2.

$$1 - (\omega_+ + \omega_-)\int_s^\infty dx \int_s^\infty dy \rho(x,y)B(x,\omega_+)B(y,\omega_-) = a(s,\omega_+)a(s,\omega_-) - b(s,\omega_+)b(s,\omega_-). \tag{5.16}$$

Remark. Using the proposition and (5.3), we can easily calculate the one-point height fluctuation.

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}[H_N(\tau=0,\beta_0)\le s] \\
&= \left(1+\frac{1}{\omega_+ +\omega_-}\frac{\partial}{\partial s}\right)\big\{a(s,\omega_+)a(s,\omega_-)-b(s,\omega_+)b(s,\omega_-)\big\}F_2(s) \\
&= a(s,\omega_+)a(s,\omega_-)F_2(s)+\frac{1}{\omega_+ +\omega_-}\big\{a(s,\omega_+)a(s,\omega_-)-b(s,\omega_+)b(s,\omega_-)\big\}F_2'(s). \qquad (5.17)
\end{aligned}$$

This corresponds to the expression of the one-point fluctuation in [27].

Proof. Let f denote the function on the left hand side. By differentiation, one finds

$$\frac{\partial}{\partial s}f(s)=(\omega_+ +\omega_-)\big\{a(s,\omega_+)a(s,\omega_-)-f(s)\big\}. \qquad (5.18)$$

This can be read as

$$f(s)=a(s,\omega_+)a(s,\omega_-)+\frac{\partial}{\partial s}\int_s^\infty dx\int_s^\infty dy\rho(x,y)B(x,\omega_+)B(y,\omega_-). \qquad (5.19)$$

The second term is now computed as

$$\begin{aligned}
&\frac{\partial}{\partial s}\int_s^\infty dx\int_s^\infty dy\rho(x,y)B(x,\omega_+)B(y,\omega_-) \\
&= -B(s,\omega_+)\int_s^\infty dy\rho(s,y)B(y,\omega_-)+\int_s^\infty dxB(x,\omega_+)\frac{\partial}{\partial s}\int_s^\infty dy\rho(x,y)B(y,\omega_-) \\
&= -B(s,\omega_+)\int_s^\infty dy\rho(s,y)B(y,\omega_-) \\
&\quad +\int_s^\infty dxB(x,\omega_+)\int_s^\infty dy\big\{-\rho(x,s)+\delta_+(x-s)\big\}\rho(y,s)B(y,\omega_-) \\
&= -\int_s^\infty dx\rho(x,s)B(x,\omega_+)\int_s^\infty dy\rho(y,s)B(y,\omega_-) \\
&= -b(s,\omega_+)b(s,\omega_-). \qquad (5.20)
\end{aligned}$$

where we use the equation (2.16) in [5]. □

6 Conclusion

We have studied the PNG model with external sources. We have represented the multi-point equal time correlation functions of the height fluctuation as the Fredholm determinant. In order to get these quantities we have first identified the limit shape of the model. The results are summarized in Theorem 2.1. The limit shape consists of a straight line and a circular one. There are three cases according to the value of $\gamma_+\gamma_-$ and they are illustrated

in Fig.2. The statistics of the one-point height fluctuation is Gaussian on the straight line and the GUE Tracy-Widom distribution on the circular line. In addition, there are special points denoted by $\beta_\pm$ where the two limit shapes meet and their statistics of the height fluctuation is GOE^2 when $\gamma_+\gamma_- < 1$ or F_0 when $\gamma_+\gamma_- = 1$.

Our main results are summarized as Theorems 3.1, 4.1 and 5.1. The spatial configuration of the height fluctuation over some region of the surface has been shown to converge to the stochastic processes represented by the Fredholm determinants obtained in the above theorems. In addition to the one-dimensional Brownian motion and the Airy process which describe the correlations near edges and in the bulk respectively, we have obtained the Fredholm determinant which describes the correlation near the points $\beta_\pm$.

These quantities vary with the parameter of the nucleation at edges $\gamma_\pm$ in (2.4) and (2.5). For the case of $\gamma_+\gamma_- < 1$ the Fredholm determinant obtained in Theorem 4.1 describes the transition between GOE^2 and GUE/Gaussian. On the other hand, when $\gamma_+\gamma_- = 1$ the points $\beta_\pm$ merge to β_0 and we have obtained the Fredholm determinant expression with the transition kernel between F_0 and Gaussian. These Fredholm determinants have the parameters τ_i which determine the position in the surface. When we set $\tau_i = \tau \neq 0$ we can obtain the one-point distribution at the point away from $\beta_\pm$. Fig.4 shows the comparison of these transitive distributions with the Monte-Carlo simulations of the PNG model in the corresponding points. These figures confirm our analyses.

Acknowledgment

The authors would like to thank M. Katori and T. Nagao for fruitful discussions and comments. They also thank the Yukawa Institute for Theoretical Physics at Kyoto University, where this study was influenced by the stimulating discussions during the YITP-W-03-18 on " Stochastic models in statistical mechanics". The work of T.I. is partly supported by the Grant-in-Aid for JSPS Fellows, the Ministry of Education, Culture, Sports, Science and Technology, Japan. The work of T.S. is partly supported by the Grant-in-Aid for Young Scientists (B), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

References

[1] P. Meakin. *Fractals, scaling and growth far from equilibrium.* Cambridge, 1998.

[2] M. Kardar, G. Parisi and Y. C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56:889–892, 1986.

[3] L.-H. Gwa and H. Spohn. Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian. *Phys. Rev. Lett.*, 68:725–728, 1992.

[4] D. Kim. Bethe ansatz solution for crossover scaling functions of the asymmetric XXZ chain and the Kardar-Parisi-Zhang-type growth model. *Phys. Rev. E*, 52:3512–3524, 1995.

[5] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Commun. Math. Phys.*, 159:151–174, 1994.

[6] C. A. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Commun. Math. Phys.*, 177:727–754, 1996.

[7] M. L. Mehta. *Random Matrices*. Academic, 2nd edition, 1991.

[8] K. Johansson. Shape fluctuations and random matrices. *Commun. Math. Phys.*, 209:437–476, 2000.

[9] M. Prähofer and H. Spohn. Current fluctuations for the totally asymmetric simple exclusion process. In V. Sidoravicius, editor, *In and out of equilibrium, vol. 51 of Progress in Probability*, pages 185–204, 2002.

[10] T. Nagao and T. Sasamoto. Asymmetric simple exclusion process and modified random matrix ensembles. cond-mat/0405321.

[11] A. Rákos and G. M. Schütz. Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process. cond-mat/0405464.

[12] J. Gravner, C. A. Tracy and H. Widom. Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Stat. Phys.*, 102:1085–1132, 2001.

[13] S. N. Majumdar and S. Nechaev. An Anisotropic Ballistic Deposition Model with Links to the Ulam Problem. and the Tracy-Widom Distribution *Phys. Rev. E*, 69: Art. No. 011103, 2004

[14] M. Prähofer and H. Spohn. Statistical self-similarity of one-dimensional growth processes. *Physica A*, 279:342–352, 2000.

[15] J. Baik, P. A. Deift and K. Johansson. On the distribution of the length of the longest increasing subsequence in a random permutation. *J. Amer. Math. Soc.*, 12:1119–1178, 1999.

[16] J. Baik and E. M. Rains. Algebraic aspects of increasing subsequences. *Duke Math. J.*, 109:1–65, 2001.

[17] J. Baik and E. M. Rains. The asymptotics of monotone subsequences of involutions. *Duke Math. J.*, 109:205–281, 2001.

[18] J. Baik and E. M. Rains. Symmetrized random permutations. In P. M. Bleher and A. R. Its, editors, *Random Matrix Models and Their Applications*, pages 1–29, 2001.

[19] M. Prähofer and H. Spohn. Universal distributions for growth processes in 1+1 dimensions and random matrices. *Phys. Rev. Lett.*, 84:4882–4885, 2000.

[20] P. L. Ferrari. Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues. math-phys/0402053.

[21] M. Prähofer and H. Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.*, 108:1071–1106, 2002.

[22] K. Johansson. Discrete polynuclear growth and determinantal processes. *Com. Math. Phys.*, 242: 277–329, 2003.

[23] P. L. Ferrari and H. Spohn. Step fluctuations for a faceted crystal *J. Stat. Phys.*, 113:1–46, 2003.

[24] P. L. Ferrari ,M. Prähofer and H. Spohn. Fluctuations of an Atomic Ledge Bordering a Crystalline Facet. *Phys. Rev. E*, 69: Art. No. 035102, 2004.

[25] T. Sasamoto and T. Imamura. Fluctuations of the One-Dimensional Polynuclear Growth Model in Half-Space *J. Stat. Phys.*, 115:749-803, 2004.

[26] T. Nagao, M. Katori and H. Tanemura. Dynamical correlations among vicious random walkers. *Phys. Lett. A*, 307:29–35, 2003.

[27] J. Baik and E. M. Rains. Limiting distributions for a polynuclear growth model with external sources. *J. Stat. Phys*, 100:523–541, 2000.

[28] K. Johansson. Non-intersecting paths, random tilings and random matrices. *Probab. Theory Relat. Fields*, 123: 225–280, 2002.

[29] F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys*, 3:1191–1198, 1962.

[30] P. J. Forrester, T. Nagao and G. Honner. Correlations for the orthogonal-unitary and symplectic transitions at the hard and soft edges. *Nucl. Phys. B*, 553:601–643, 1999.

[31] A. M. S. Macêdo. Universal parametric correlations at the soft edge of spectrum of random matrix ensembles. *Europhys. Lett.*, 26:641–646, 1994.

[32] P. J. Forrester. Painlevé transcendent evaluation of the scaled distribution of the smallest eigenvalue in the Laguerre orthogonal and symplectic ensembles. nlin.SI/0005064.

[33] M. Prähofer and H. Spohn. Exact scaling functions for one-dimensional stationary KPZ growth. *J. Stat. Phys.*, 115:255–279. 2004.

[34] T. Imamura and T. Sasamoto. in progress

[35] C. A. Tracy and H. Widom. The distribution of the largest eigenvalue in the Gaussian ensembles: $\beta = 1, 2, 4$. *CRM Ser. Math. Phys.*, 461-472. 2000.

[36] C. A. Tracy and H. Widom. Universality of the distribution functions of random matrix theory, In M. T. Batchelor and L. T. Wille, editors, *Statistical Physics on the Eve of the 21st Century: In Honor of J B McGuire on the Occasion of His 65th Birthday*, pages 230–239, 1999.

Figure Captions

Fig.1: Rules of the discrete PNG model. Fig. (a) shows an example of the rules 1 and 2. A step with height k at the origin is generated according to the geometric distribution. It grows laterally one step in both directions in the next time step.

Fig. (b) shows a collision of the steps with height one and two. The height of the origin, which is the point they collide, is two by the rule 3.

Fig.2: Typical examples of the limit shape of the PNG model with external sources. The solid lines show the snapshots of Monte Carlo simulations for $N = 5000, \alpha = 0.32$. The dashed lines represent the limit shape in bulk given by (2.6). The parameters γ_+ and γ_- are $\gamma_+ = 0.79, \gamma_- = 0.63$ in (a), $\gamma_+ = 1.58, \gamma_- = 1.26$ in (b) and $\gamma_+ = 1.58, \gamma_- = 0.63$ in (c).

Fig.(a) corresponds to the region (i) in Theorem 2.1 where the hight fluctuation in bulk obeys the GUE Tracy-Widom distribution while the fluctuations near both edges are described by Gaussian. Fig.(b) corresponds to the region (ii) where the height fluctuations at all points obey the Gaussian distribution. Fig.(c) represent the case of region (iii). The limit shape is a straight line tangent to the limit shape in bulk.

Fig. 3: Average in the GOE^2 to GUE/Gaussian transition. Three lines represent the asymptotics in the case where $\tau \sim \infty, 0, -\infty$ respectively. $+$ gives the value obtained by solving (4.23)–(4.26) numerically.

Fig4: The limiting distribution near the competing points between bulk and edge. In Fig.(a) the three curves are the distributions for $\tau = 1, 0, -1$ from the left obtained in Theorem 4.1. Each point is the data of a Monte-Carlo simulation near the origin where $t = 2000(N = 1000), \alpha = 0.32, \gamma_+ = 0.32, \gamma_- = 1.0$ and $x = 352, 0, -352$ respectively. Fig.(b) shows the distribution near F_0. The curves represent the transitive distributions for $\tau = 0, 1$ from the left using Theorem5.1. The points are numerical data for $t = 2000(N = 1000), \alpha = 0.32, \gamma_+ = 1.0, \gamma_- = 1.0$ and $x = 0, 352$ respectively.

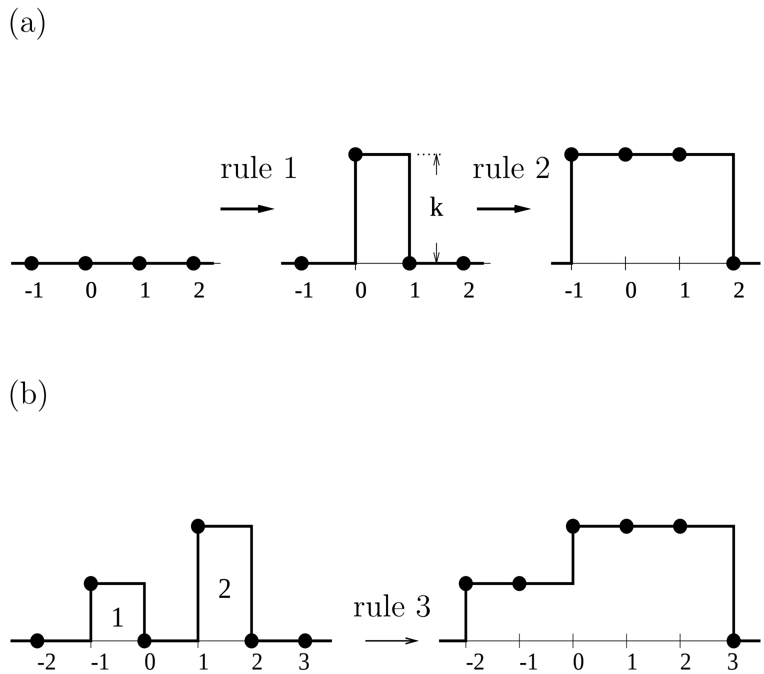


Figure 1

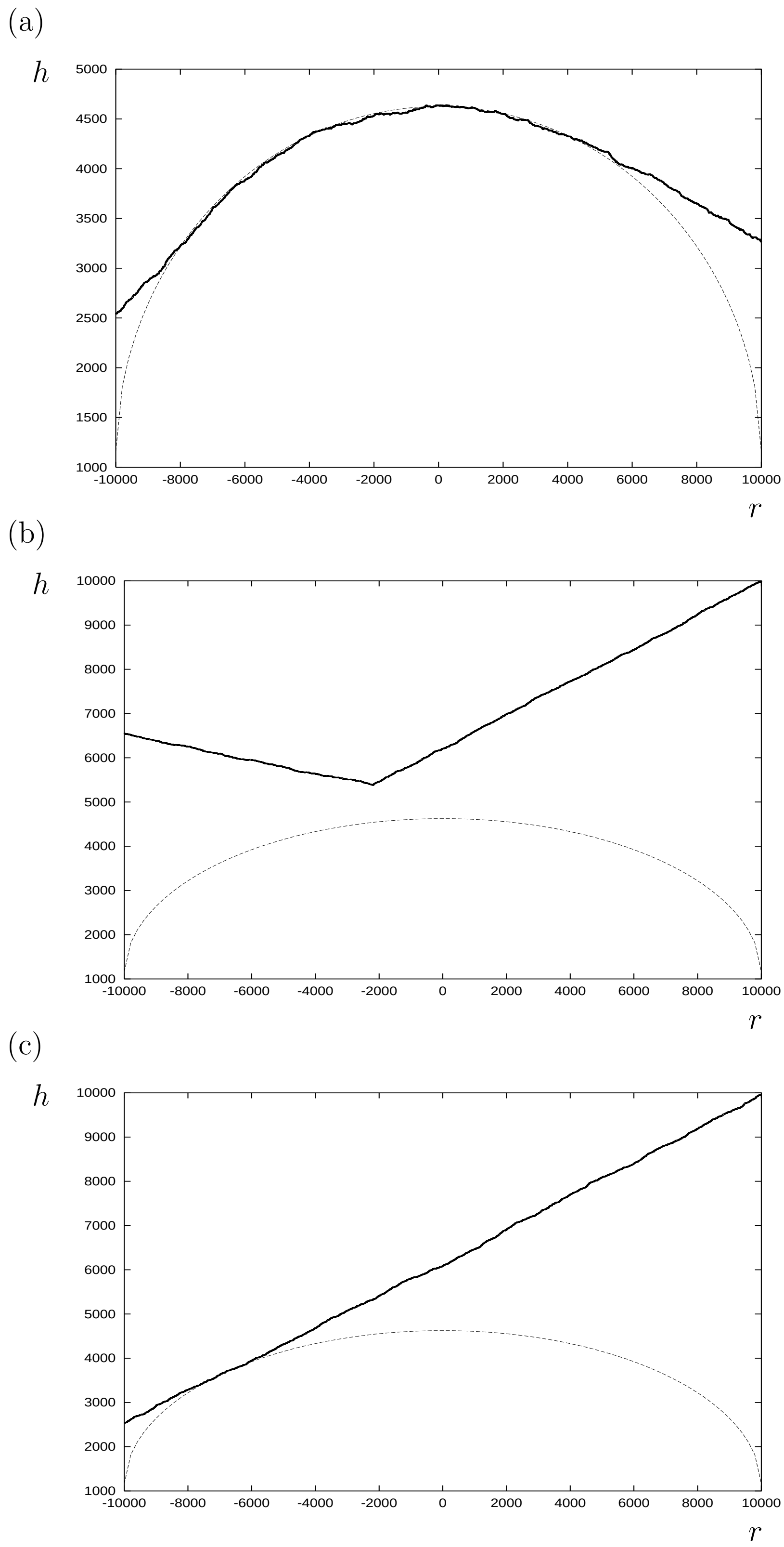
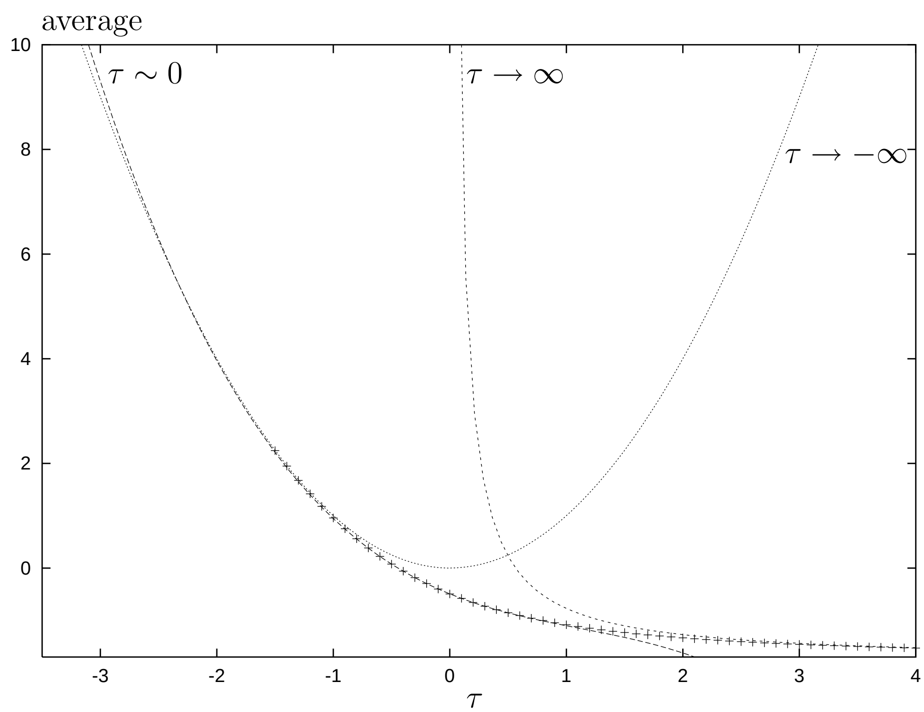


Figure 2



average
10
8
6
4
2
0
$\tau \sim 0$
$\tau \to \infty$
$\tau \to -\infty$
-3
-2
-1
0
1
2
3
4
τ

Figure 3

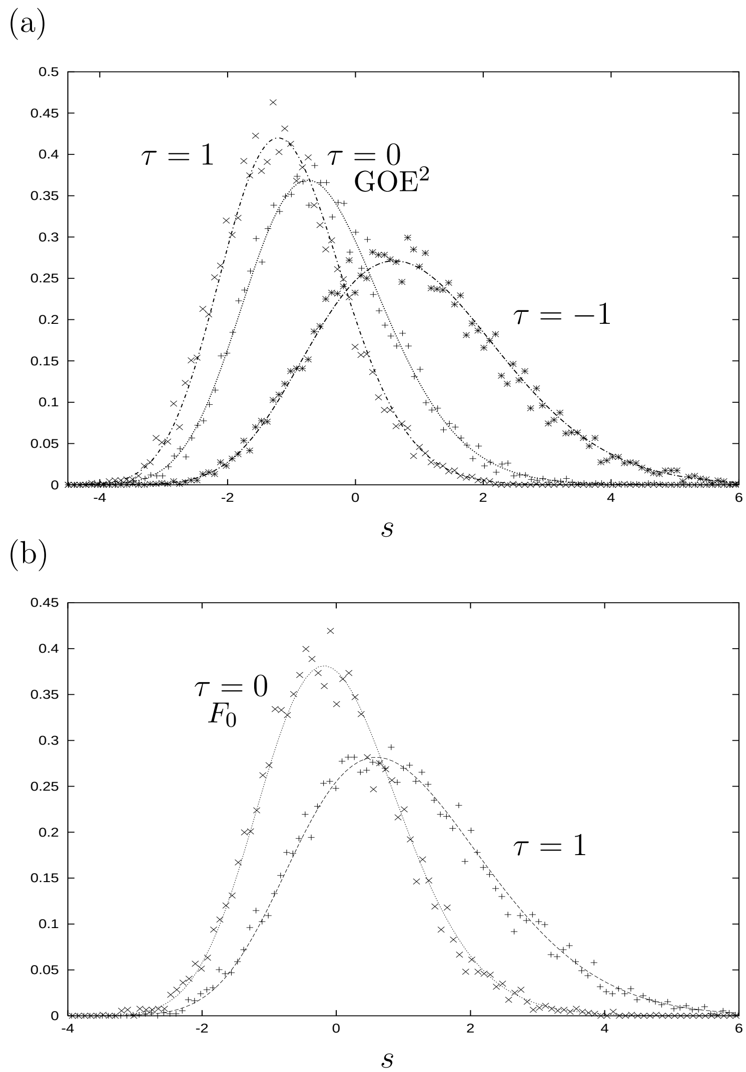


Figure 4